Symplectic AV-bundle reduction

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Abstract. An AV-bundle $\mu : A \to V$ is just an $\mathbb{R}$-principal bundle. If A is a symplectic manifold and the principal action is symplectic, μ is said to be *symplectic*. Symplectic AV-bundles play an important role in the geometric formulation of non-autonomous Hamiltonian systems. Given a symplectic AV-bundle $\mu : A \to V$ with a compatible symplectic action of a Lie group G on A, such that some regularity and compatibility properties hold, we obtain a reduced symplectic AV-bundle $\mu_\nu : A_\nu \to V_\nu$. We show also that any G-invariant Hamiltonian section induces a reduced Hamiltonian section of μ_ν. Mardsen-Weinstein reduction theory for symplectic, cosymplectic and Poisson manifolds is used.

1. Symplectic AV-bundles and non-autonomous Hamiltonian systems

An AV-*bundle* [3] is just a $\mathbb{R}$-principal bundle $\mu : A \to V$. We will denote by

$$\psi : \mathbb{R} \times A \to A, \qquad (t, a) \mapsto \psi_t(a),$$

the associated principal action of the Lie group $(\mathbb{R}, +)$ on the manifold A and by $Z_\mu \in \mathfrak{X}(A)$ the infinitesimal generator of ψ.

Definition 1 We will say that $\mu : (A, \Omega) \to V$ is a *symplectic AV-bundle*, if $\mu : A \to V$ is an AV-bundle, (A, Ω) is a symplectic manifold and the associated principal action is *symplectic*.

Example 1 Let $\pi : M \to \mathbb{R}$ be a surjective submersion. If Ω_M is the canonical symplectic form on T^*M and $V^*\pi$ is the dual bundle of the vertical bundle of π, then the projection $\mu_\pi : (T^*M, \Omega_M) \to V^*\pi$ is a symplectic AV-bundle (*the standard symplectic AV-bundle*).

Proposition 1 ([4]) *Let $\mu : (A, \Omega) \to V$ be a symplectic AV-bundle. Then there exists a unique Poisson structure $\{\cdot,\cdot\}_V$ on V such that μ is a Poisson map, i.e.*

$$\{f \circ \mu, f' \circ \mu\}_A = \{f, f'\}_V \circ \mu, \qquad \text{for any } f, f' \in C^\infty(V),$$

where $\{\cdot,\cdot\}_A$ is the Poisson bracket on A induced by Ω.

Definition 2 A *non-autonomous Hamiltonian system* (A, μ, Ω, h) is a symplectic AV-bundle $\mu : (A, \Omega) \to V$ endowed with a section $h : V \to A$ of μ, i.e. a smooth map such that $\mu \circ h = id_V$. The section $h : V \to A$ is called the *Hamiltonian section of the system*.

There is a one-to-one correspondence between the sets:

$$\{h : V \to A |\, h \text{ section of } \mu\} \leftrightarrow \{F_h \in C^\infty(A) |\, Z_\mu(F) = 1\}$$

defined by the following relation

$$a = \psi\left(F_h(a), h(\mu(a))\right), \qquad \text{for any } a \in A.$$

Theorem 1 ([4]) *Let (A, μ, Ω, h) be a non-autonomous Hamiltonian system with infinitesimal generator Z_μ. If $\omega_h \in \Omega^2(V)$ and $\eta_h \in \Omega^1(V)$ are the forms defined as*

$$\omega_h = h^*\Omega, \qquad \eta_h = -h^*(i_{Z_\mu}\Omega),$$

then (V, ω_h, η_h) is a cosymplectic manifold. The Reeb vector field $\mathcal{R}_h \in \mathfrak{X}(V)$ is just the μ-projection of the Hamiltonan vector field $\mathcal{H}_{F_h}$ of F_h.
Moreover, the Poisson bracket on V induced by (ω_h, η_h) is just the Poisson bracket $\{\cdot,\cdot\}_V$.

Let (A, μ, Ω, h) be a non-autonomous Hamiltonian system. Then, we can choose Darboux coordinates (t, p, q^i, p_i) on A such that the local expression of $\mu : A \to V$ is

$$\mu(t, p, q^i, p_i) = (t, q^i, p_i).$$

Thus, if the local espression of $h : V \to A$ is

$$h(t, q^i, p_i) = (t, -H(t, q^j, p_j), q^i, p_i),$$

then, given a curve γ on V

γ is an integral curve of R_h $\iff$ $\gamma(t) = (t, q^i(t), p_i(t))$ is a solution of the *Hamilton equations* for H.

We will say that the vector field $\mathcal{R}_h$ describes the *dynamics* of the non-autonomous Hamiltonian system.

2. AV-bundle reduction Theorem

Definition 3 An action $\phi : G \times A \to A$ is said to be a *canonical action* on the symplectic AV-bundle $\mu : (A, \Omega) \to V$ if the following conditions hold:

i) ϕ is a symplectic action;

ii) the actions ψ and ϕ commute, that is $\phi_g \circ \psi_t = \psi_t \circ \phi_g$, for any $g \in G$, $t \in \mathbb{R}$;

iii) the 1-form $\theta_\mu = i_{Z_\mu}\Omega$ is basic with respect to ϕ, i.e. $\theta_\mu(\xi_A) = 0$ for any $\xi \in \mathfrak{g}$, where ξ_A is the infinitesimal generator of ϕ defined by ξ.

By passing to the quotient, we have

If ϕ is a canonical action and $J : A \to \mathfrak{g}^*$ is an equivariant momentum map $\implies$ $\phi^V : G \times V \to V$ a Poisson action on $(V, \{\cdot,\cdot\}_V)$, $J^V : V \to \mathfrak{g}^*$ an equivariant momentum map (with respect to ϕ^V)

Suppose that ϕ^V is free and proper and denote by G_ν the isotropy group of an element $\nu \in \mathfrak{g}^*$ with respect to the coadjoint action of G. Denote by A_ν and V_ν the quotient manifolds given by

$$A_\nu = J^{-1}(\nu)/G_\nu, \qquad V_\nu = (J^V)^{-1}(\nu)/G_\nu.$$

By passing again to the quotient, we may obtain

$\mu : J^{-1}(\nu) \to (J^V)^{-1}(\nu)$ is G_ν-invariant with principal action $\psi : \mathbb{R} \times J^{-1}(\nu) \to J^{-1}(\nu)$ $\implies$ $\mu_\nu : A_\nu \to V_\nu$ a surjective submersion and $\psi_\nu : \mathbb{R} \times A_\nu \to A_\nu$ a principal action

Marsden-Weinstein reduction [5]

$$\begin{tikzcd}
(A, \Omega) \arrow[r, "\mu"] \arrow[d, dashed] & (V, \{\cdot,\cdot\}) \arrow[d, dashed] \\
(A_\nu, \Omega_\nu) \arrow[r, "\mu_\nu"] & (V_\nu, \{\cdot,\cdot\}_\nu)
\end{tikzcd}$$

Poisson reduction [6]

Theorem 2 ([4]) *Let $\mu : (A, \Omega) \to V$ be a symplectic AV-bundle equipped with a canonical action $\phi : G \times A \to A$ and an equivariant momentum map $J : A \to \mathfrak{g}^*$. Suppose that the induced action $\phi^V : G \times V \to V$ is free and proper. Then, for any $\nu \in \mathfrak{g}^*$, $\mu_\nu : (A_\nu, \Omega_\nu) \to V_\nu$ is a symplectic AV-bundle with $\mathbb{R}$-principal action $\psi_\nu : \mathbb{R} \times A_\nu \to A_\nu$. Moreover, the restriction of the infinitesimal generator Z_μ of μ to $J^{-1}(\nu)$ is tangent to $J^{-1}(\nu)$ and π_ν-projectable. Its π_ν-projection is the infinitesimal generator Z_{μ_ν} of μ_ν.*
In addition, the reduced Poisson bracket $\{\cdot,\cdot\}_\nu$ on V_ν is just the one induced by the symplectic AV-bundle $\mu_\nu : A_\nu \to V_\nu$.

3. Hamiltonian reduction Theorem

Let $\phi : G \times A \to A$ be a canonical action on the total space A of a non-autonomous Hamiltonian system. Denote by $\phi^V : G \times V \to V$ the corresponding action on V.

Definition 4 The Hamiltonian section h is said to be G-*invariant* if h is equivariant with respect to the actions ϕ and ϕ^V, that is

$$h \circ \phi^V_g = \phi_g \circ h, \qquad \text{for any } g \in G.$$

We have that

ϕ a canonical action, $J : M \to \mathfrak{g}^*$ a momentum map, $h : V \to A$ a G-invariant Hamiltonian section $\implies$ $\phi^V : G \times V \to V$ a cosymplectic action on (V, ω_h, η_h), $R_h(J_\xi) = 0$ for any $\xi \in \mathfrak{g}$

Marsden-Weinstein reduction [5]

$$\begin{tikzcd}
(A, \Omega) \arrow[r, "\mu"] \arrow[d, dashed] & (V, \omega_h, \eta_h) \arrow[d, dashed] \\
(A_\nu, \Omega_\nu) \arrow[r, "\mu_\nu"] & (V_\nu, (\omega_h)_\nu, (\eta_h)_\nu)
\end{tikzcd}$$

Cosymplectic reduction [1]

Moreover, if ϕ^V is free and proper and $\nu \in \mathfrak{g}^*$, we have

ϕ a canonical action, $h : V \to A$ a G-invariant Hamiltonian section $\implies$ $h_\nu : V_\nu \to A_\nu$ a Hamiltonian section of the reduced symplectic AV-bundle $\mu_\nu : (A_\nu, \Omega_\nu) \to V_\nu$

Theorem 3 ([4]) *Let (A, μ, Ω, h) be a non-autonomous Hamiltonian system and $\phi : G \times A \to A$ be a canonical action such that the induced action on V is free and proper. Suppose that $J : A \to \mathfrak{g}^*$ is an equivariant momentum map. If h is G-invariant, then, for any $\nu \in \mathfrak{g}^*$, the cosymplectic structure on V_ν induced by the non-autonomous Hamiltonian system $(A_\nu, \mu_\nu, \Omega_\nu, h_\nu)$ is just the reduced cosymplectic structure obtained from (M, ω_h, η_h).*
Moreover, the dynamics $\mathcal{R}_{h_\nu}$ of the reduced non-autonomous Hamiltonian system is just the projection of the dynamics R_h of (A, μ, Ω, h).

Conclusions and future work

We extend the classical procedure of reduction of symplectic manifolds, due to Marsden and Weinstein (see [5]), to the non-autonomous framework. If we have a non-autonomous Hamiltonian system with a symmetry, represented by a symplectic AV-bundle with a canonical action and an invariant Hamiltonian section, we obtain a reduced non-autonomous Hamiltonian system.

In particular, we can apply this procedure to the *standard symplectic AV-bundle* $\mu_\pi : T^*M \to V^*\pi$, obtained from a fibration $\pi : M \to \mathbb{R}$. It would be interesting to discuss when the reduced AV-bundle obtained from a standard AV-bundle is again standard. For this purpose, a suitable generalization of the cotangent bundle reduction could be used (see [4]).

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