Intersection cohomology of coisotropic submanifolds

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Work in progress.

Koszul complex

Let $A \to M$ be a vector bundle.

Proposition

1. $(\Gamma(\wedge^\bullet A), \wedge)$ is an algebra w.r.t. wedge product $\wedge$.
2. $(\Gamma(\wedge^\bullet A^*), \imath)$ is an module over this algebra, w.r.t. the contraction operator $\imath$.

Choose now a section $x \in \Gamma(A^*)$, and call by m_x (resp $\imath_x$) the operators

$$\Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^{\bullet+1} A^*)(\text{resp. } \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet-1} A))$$

of multiplication by x (resp. contraction by x). Both operators square to zero, hence define a cohomology $H^*(x)$ and an homology $H_*(x)$, said to be attached to x.

Proposition

1. The homology $H_*(x)$ is an algebra.
2. The cohomology $H^*(x)$ is a module over the algebra $H_*(x)$.

The differential of Lie algebroid

Definition

Let $A \to M$ be a vector bundle. A *pré-Lie algebroid structure* on A is a graded derivation of degree $+1$:

$$D : \Gamma(\wedge^{\bullet} A^*) \to \Gamma(\wedge^{\bullet+1} A^*).$$

It is said to be a *Lie algebroid* when it squares to 0.

The *Lie derivative* w.r.t. $P \in \Gamma(\wedge^k A)$ is the operator defined by:

$$\mathcal{L}_P := [\imath_P, D],$$

The *bracket* of $P \in \Gamma(\wedge^k A)$, $Q \in \Gamma(\wedge^l A)$ is the unique element $R := [P, Q]_D \in \Gamma(\wedge^{k+l-1} A)$ s.t.

$$\begin{cases} [\mathcal{L}_P, \mathcal{L}_Q] & = & \mathcal{L}_R, \\ [\mathcal{L}_P, \imath_Q] & = & \imath_R. \end{cases}$$

The differential of Lie algebroid

A *Gerstenhaber algebra* is a graded vector space E, endowed with a graded commutative product, an a graded Lie algebra structure on $\overline{E}$, with are compatible in the sense that:

$$[P, QR] = [P, Q]R + (-1)^{qr}[P, R]Q.$$

A *Gerstenhaber algebra module* is a graded vector space which is a module for both the commutative product and the Lie bracket + compatibility relations.

When there is no Jacobi identity, we speak of *pré-Gerstenhaber algebra, pré-Gerstenhaber module*.

Example : Lie algebra of multivector fields, and exterior forms.

Proposition

Let $A \to M$ be a vector bundle. For every (pré)-algebroid structure D, the triple $(\Gamma(\bigwedge^\bullet A), \wedge, [.,.]_D)$ is a (pré)-Gerstenhaber algebra, and $\Gamma(\bigwedge^\bullet A^*)$ is a module over this (pré)-Gerstenhaber algebra.

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Induced structure on Koszul complex.

Question 1. Let a vector bundle $A \to M$. Given:

1. a section $x \in \Gamma(A^*)$
2. a (pré)-Lie algebroid D,

do the (pré)-Gerstenhaber algebra $(\Gamma(\bigwedge^\bullet A), \wedge, [.,.]_D)$, and its module $\Gamma(\bigwedge^\bullet A^*)$ go to the quotient with respect to $m_x, \imath_x$ and define (pré)-Gerstenhaber algebra structures and modules on $H_*(x), H^*(x)$ respectively ???

Proposition

The answer is **yes** if and only if $D(x) = 0$.

Question 2. But then, could it be that D be a pré-Lie algebroid but still the induced structure is a Gerstenhaber algebra ???

Induced structure on Koszul complex-2.

Theorem

Let $A \to M$ be a vector bundle. Given:

1. a section $x \in \Gamma(A^*)$
2. a pré-Lie algebroid D,

such that

- ♣ $D(x) = 0$
- ♠ $D^2 = C \circ m_x - m_x \circ C$ for some operator C (i.e. homotopic to zero),

then the homologies $H_*(x)$ and co-homologies $H^*(x)$ attached to x admits induced Gerstenhaber algebra structures and modules respectively.

(And there is no "pré" anymore in the conclusion !)

Coisotropic submanifolds

Let (X,π) be a Poisson manifold. A submanifold $M \subset X$ is said to be *coisotropic* if one of the equivalent conditions is satisfied:

1. $\pi^{\#}(T_mM^{\perp}) \subset T_mM$
2. the ideal of functions vanishing on M is closed under Poisson bracket,
3. in a local adapted system of coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_d)$, the Poisson structure is of the form
$$\pi = \sum_{i,j} a_{i,j}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j} + \sum_{i,k} b_{i,k}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial x_k} + \sum_{k,l} c_{k,l}\frac{\partial}{\partial x_k}\wedge\frac{\partial}{\partial x_l},$$
where $c_{k,l}$ is identically zero on M.

Facts

1. (X,π) Poisson $\Rightarrow$ T^*X Lie algebroid, $\Rightarrow$ the operator $[\pi,\cdot]$ is a derivation squaring to 0 of $\Gamma(\wedge^\bullet TX)$.
2. $M \subset X$ coisotropic $\Rightarrow$ $TM^{\perp}$ Lie sub-algebroid, $\Rightarrow$ induces a derivation squaring to 0 of $\Gamma(\wedge^\bullet TX/TM)$.

L_∞-algebras

Definition

An L_∞-algebra is a graded vector space endowed with a co-derivation $\mathcal{D}$ of degree $+1$ squaring zero on $S^\bullet(\overline{E})$ (considered as a co-algebra w.r.t. de-concatenation).

Meaning. A co-derivation is entirely determined by its components $[\cdots]_n : S^n\overline{E} \to \overline{E}$ which are skew-symmetric maps mapping $E_{i_1} \times \cdots \times E_{i_n}$ to $E_{i_1+\cdots+i_n-n+2}$.

Example. Lie algebras, complexes and Differential Graded Lie Algebras (DGLA).

Definition

A Maurer-Cartan element is an element $x \in E_1$ such that $\mathcal{D}(e^x) = 0$, or, equivalently:

$$\sum_{n\in\mathbb{N}} \frac{[x,\ldots,x]_n}{n!} = 0.$$

Definition

An L_∞-algebra structure on a graded vector space $\sum_n E_n$ is a sequence of operations:

$$[E_{i_1}, \dots, E_{i_n}] \mapsto E_{i_1 + \dots + i_n - n + 2},$$

which are skew-symmetric in each variables, and satisfy some (generalized) Jacobi identity:

$$\sum_{i+j=n} \sum_{\sigma \in \mathrm{Schuffles}(i, i+j-1)} \varepsilon(\sigma)[[x_{\sigma(1)}, \dots, x_{\sigma(i)}]_i, x_{\sigma(i+1)}, \dots, x_{\sigma(i+j-1)}]_j$$

Example of D.G.L.A. The Jacobi identity amounts to

1. $D^2 = 0$,
2. D is a derivation of $[\cdot, \cdot]$,
3. and $[\cdot, \cdot]$ satisfies the Jacobi identity.

L_∞-algebras and coisotropic submanifolds

Important Example: Given

1. a Poisson manifold (X, π),
2. a coisotropic submanifold M,
3. global transverse coordinates (*i.e.* a diffeomorphism between a neighborhood of the zero section in TX/TM and a neighborhood of M in X, *i.e.* a normal bundle and linear coordinates on it),

there is an induced L_∞ structure on $\Gamma(\wedge^\bullet TX/TM)$, constructed as follows:

$$[P_1, \ldots, P_n]_n := p(\left[\left[\pi, \widehat{P_1}\right], \ldots, \widehat{P_n}\right])$$

where $\widehat{P_1}, \ldots, \widehat{P_n}$ are the unique multivector vector fields invariant by translation along the fibers that extend $P_1, \ldots, P_n$, and p is the operator that takes the projection on $\Gamma(\wedge^\bullet TX/TM)$ of a multivector field on X.

Maurer-Cartan

Proposition

Assume that the Poisson structure is analytic. A section $x \in \Gamma(TX/TM)$ is Maurer-Cartan if and only if the time-1 flow of $\widehat{x}$ maps M to a coisotropic submanifold.

Said differently, Maurer-Cartan equation encodes deformations of coisotropic submanifolds. For non-analytic submanifolds, the result is only true formally.

Example: Given two coisotropic submanifolds M, N of the same dimension of (X, π), there exists adapted coordinates $(y_1, \dots, y_n, x_1, \dots, x_d)$ for M such that N is given by $x_i := X_i(y_1, \dots, y_n)$. Then

$$x := \sum_n X_i \frac{\partial}{\partial x_i}$$

is a Maurer-Cartan element w.r.t. the L_∞-structure (when it is seen as a section of $\Gamma(TX/TM)$).

Maurer-Cartan

In short. Given two coisotropic submanifolds M, N of the same dimension in an analytic Poisson manifolds (X, π), there exists, near any intersection point,

1. a L_∞-structure $([\cdots]_n)_{n\in\mathbb{N}_*}$ on the graded algebra $\Gamma(\wedge^\bullet TX/TM)$,
2. a Maurer-Cartan element.

Question 1. Is it canonical ? No, but other choices would lead to "equivalent" data. (We leave it aside).

Question 2. What else ? The L_∞ structure is indeed a P_∞ structure, i.e. the brackets $([\cdots]_n)_{n\in\mathbb{N}^*}$ are graded derivations in each argument, when $\Gamma(\wedge^\bullet TX/TM)$ is equipped with the wedge product.

The fact that the L_∞-structure is indeed a P_∞ structure matters (because on P_∞ structures, there is a notion of derivation of degree $+1$):

Proposition

Let x be a Maurer-Cartan element of a P_∞ structure, then the operator

$$\delta(y) := \sum_{n\geq 1} \frac{1}{n!}[x, \ldots, x, y]_n$$

(assumed to be converging) is a derivation of degree $+1$ and it satisfies the condition ♣, i.e $D(x) = 0$.

Question. Can we manage to have also condition ♠ ? Is $D^2 = 0$??? No. Is D^2 homotopic to zero ???

Well-known fact : Given a Maurer-Cartan element, the operator:

$$D_x : y \mapsto \sum_{n\geq 1} \frac{1}{(n-1)!}[x, \ldots, x, y]_n$$

From P_∞ to Gerstehanber

Theorem

Let x be a Maurer-Cartan element of a P_∞ structure, then the operator

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(assumed to be converging) is a derivation of degree $+1$ and it satisfies the condition ♣, i.e $D(x) = 0$. If, moreover, the P_∞-structure is quantizable (by deformation) in an A_∞-structure whose term in ε is skew-symmetric, then the condition ♠ is also satisfied.

Indeed, it suffices that the Hochshield cohomology of the underlying algebra is concentrated in skew-symmetric part.

Example: The P_∞-structure associated to a coisotropic submanifold satisfies all these assumptions [Cattaneo-Felder].

Conclusion

General idea:

1. X Poisson + M coisotropic $\rightsquigarrow$ P_∞-structure on $\Gamma(\wedge^\bullet TX/TM)$.
2. N coisotropic $\rightsquigarrow$ Maurer-Cartan element $x \in \Gamma(TX/TM)$.
3. Previous theorem $\rightsquigarrow$ Gerstenhaber algebra structure on $H_*(x)$ and Gerstenhaber module on $H^*(x)$.

More precisely:

Corollary

Let (X, π) be an analytic Poisson manifold. Let M, N be a coisotropic submanifolds, and x the a Maurer-Cartan element on the P_∞ structure on $\Gamma(\wedge^\bullet TX/TM)$, constructed with the help of some well-chosen coordinates on a neighborhood U of a point in $M \cap N$. Let $x \in \Gamma(TX/TM)$ be the Maurer-Cartan solution corresponding to it, then then the homologies $H_*(x)$ and co-homologies $H^*(x)$ attached to x admits induced Gerstenhaber algebra structures and modules respectively..

Conclusion

Black box. These structures do not depend on the several choices made in the construction, and can be glued together in a global object.

Let us admit it. Now, what does $H^*(x)$ and $H_*(x)$ *really* compute ?

Proposition

Given two submanifolds M, N on X, and a coordinate neighborhood $\mathcal{U}$ adapted to M, in which N is the graph of some map, that we can see as a section $x \in \Gamma_{\mathcal{U}\cap M}(TX/TM)$. Then $H_*(x) = \mathrm{Tor}_U(M, N)$ and $H^*(x) = \mathrm{Ext}_U(M, N)$, i.e $Tor_{\mathcal{F}(\mathcal{U})}(\mathcal{F}(M \cap \mathcal{U}), \mathcal{F}(N \cap \mathcal{U}))$ and $Ext_{\mathcal{F}(\mathcal{U})}(\mathcal{F}(M \cap \mathcal{U}), \mathcal{F}(N \cap \mathcal{U}))$ respectively.

Question. Baranovski-Ginzburg (following Behrend-Fanteschi) have constructed a Gerstenhaber algebra structure on the sheafified *Tor* and *Ext* of coisotropic submanifolds of an algebraic Poisson manifold. Does our construction matches ? **Yes** in the symplectic case.