Marsden-Weinstein reduction theory for the symplectic prolongation of a Lie algebroid

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Motivation and problems

Marsden-Weinstein reduction theorem for symplectic manifolds (Marsden, Weinstein 1974)

Let (M, Ω) be a symplectic manifold and $\phi : G \times M \to M$ be a hamiltonian action of G on M with equivariant momentum map $J : M \to \mathfrak{g}^*$. Suppose that $\mu \in \mathfrak{g}^*$ is a regular value of $J : M \to \mathfrak{g}^*$ and that the space of orbits $J^{-1}(\mu)/G_\mu$ of the action of G_μ on $J^{-1}(\mu)$ is a quotient manifold. Then $M_\mu = J^{-1}(\mu)/G_\mu$ is a symplectic manifold with symplectic form Ω_μ which is characterized by the condition

$$\pi_\mu^*(\Omega_\mu) = \iota_\mu^*(\Omega),$$

where $\pi_\mu : J^{-1}(\mu) \to M_\mu$ is the canonical projection and $\iota_\mu : J^{-1}(\mu) \to M$ is the canonical inclusion.

Motivation and problems

Marsden-Weinstein reduction theorem for Poisson manifolds (Marsden, Ratiu 1986)

Let $(P, \{\cdot,\cdot\})$ be a Poisson manifold and $\phi : G \times P \to P$ be a hamiltonian action of G on P with equivariant momentum map $J : P \to \mathfrak{g}^*$. Suppose that $\mu \in \mathfrak{g}^*$ is a regular value of $J : P \to \mathfrak{g}^*$ and that the space of orbits $J^{-1}(\mu)/G_\mu$ of the action of G_μ on $J^{-1}(\mu)$ is a quotient manifold. Then $P_\mu = J^{-1}(\mu)/G_\mu$ is a Poisson manifold with Poisson bracket $\{\cdot,\cdot\}_\mu$ which is characterized by the condition

$$\{\tilde{f}, \tilde{g}\}_\mu \circ \pi_\mu = \{f, g\} \circ \iota_\mu$$

for $\tilde{f}, \tilde{g} \in C^\infty(J^{-1}\mu/G_\mu)$, where $\pi_\mu : J^{-1}(\mu) \to P_\mu$ is the canonical projection, $\iota_\mu : J^{-1}(\mu) \to P$ is the canonical inclusion and $f, g \in C^\infty(P)$ are G-invariant extensions of $\tilde{f} \circ \pi_\mu$ and $\tilde{g} \circ \pi_\mu$, respectively.

Motivation and problems

A problem (Fernandes, Ortega, Ratiu, 2009)

Let $\Phi : G \times P \to P$ be a smooth action of a Lie group G on a Poisson manifold $(P, \{\cdot, \cdot\})$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ its dual. We assume that this is a Poisson action i.e., G acts by Poisson diffeomorphisms. Such a Poisson action will not usually have a momentum map in the classical sense.

There are, at least, three alternatives:

- A Poisson action of G on P always lifts to a symplectic action of G on the symplectic groupoid $\Sigma(P)$ which admits an equivariant momentum map (Fernandes, Ortega, Ratiu, 2009)
- To give a more suitable definition of a momentum map for a Poisson action
- $\surd$ To find a class of interesting Poisson actions which admit natural equivariant momentum maps

Motivation and problems

Certain types of linear Poisson actions on linear Poisson manifolds

We will consider the following objects:

- A vector bundle A^* endowed with a linear Poisson structure
- A Poisson action of a Lie group G on A^* "by complete lifts"

Motivation and problems

The aim of this talk

- To show that an action of a Lie group G by complete lifts on a linear Poisson manifold A^* admits a natural equivariant momentum map
- To prove that the linear Poisson structure on A^* is covered by a symplectic Lie algebroid
- To prove that the reduction of the linear Poisson structure is (under certain regularity conditions) covered by a (reduced) symplectic Lie algebroid (for this purpose, a Marsden-Weinstein reduction theorem for a general symplectic Lie algebroid will be discussed)
- To show under what conditions the reduction of a standard symplectic Lie algebroid is again a standard symplectic Lie algebroid

Collaboration with some people: D Iglesias, M de León, D Martín, E Martínez, E Padrón, M Rodríguez-Olmos,...

Plain of the talk

- Actions by complete lifts on linear Poisson manifolds
- The symplectic cover of a linear Poisson manifold
- Marsden-Weinstein reduction of symplectic Lie algebroids and linear Poisson structures
- Marsden-Weinstein reduction of the standard symplectic Lie algebroid
- Future work

Remarks Some parts of my talk are closely related with the previous talk by E Martínez:

- The symplectic cover of a linear Poisson structure on the dual bundle of a vector bundle A is just the A-tangent bundle to A^*. It is a symplectic Lie algebroid and it was used by E. Martínez in the geometric formulation of hamiltonian Mechanics on Lie algebroids
- The standard symplectic Lie algebroid is just the A-tangent bundle to A^* (where A^* is a linear Poisson manifold). We will recover the last result in the talk by E Martínez (on reduction of the standard symplectic Lie algebroid) as a corollary of our Marsden-Weinstein reduction theorem for arbitrary symplectic Lie algebroids

A vector bundle $\tau_{A^*} : A^* \to Q$ over Q of rank n

$\{\cdot,\cdot\}_{A^*}$ a linear Poisson structure on A^*

$$\Downarrow$$

$F, G : A^* \to \mathbb{R}$ linear functions on $A^* \Longrightarrow \{F, G\}_{A^*}$ is a linear function on A^*

Remark: $\{F : A^* \to \mathbb{R} / F \text{ is linear }\} \Longleftrightarrow \Gamma(A)$

$$X \in \Gamma(A) \Longrightarrow \hat{X} : A^* \to \mathbb{R},\ \hat{X}(\alpha) = \alpha(X(\tau_{A^*}(\alpha)))$$

Interesting examples of linear Poisson manifolds for dynamics

- $A^* = T^*Q$ the dual bundle of TQ endowed with the Poisson structure induced by the canonical symplectic structure; standard Hamiltonian dynamics
- $A^* = \mathfrak{g}^*$ the dual space of a real Lie algebra $\mathfrak{g}$ endowed with the Lie-Poisson structure; Lie-Poisson dynamics (Rigid bodies)
- $A^* = T^*Q/G$ the dual bundle of the quotient vector bundle TQ/G (here, $\pi : Q \to Q/G$ is a principal G-bundle) endowed with the Weinstein space Poisson bracket; Hamilton-Poincaré dynamics (reduction of hamiltonian systems which are invariant under the action of a Lie group of symmetries)

Interesting examples of linear Poisson manifolds for dynamics

- $A^* = Q \times \mathfrak{g}^*$ as a vector bundle over Q (here, $\mathfrak{g}$ is a Lie algebra and the linear Poisson structure on A^* is induced by an infinitesimal action of $\mathfrak{g}$ on Q); reduction of hamiltonian systems and homogeneous spaces (the heavy top)
- $A^* = \dfrac{V^*\pi}{G}$ the dual bundle of the quotient vector bundle $\dfrac{V\pi}{G}$ (here, $\pi : Q \to \mathbb{R}$ is a fibration, $V\pi$ is the vertical bundle and the action of G on Q is free, proper and π-fibered); reduction of non-autonomous hamiltonian systems

$\{\cdot,\cdot\}_{A^*}$ a linear Poisson bracket on A^*

$$\Downarrow$$

- $\exists [\![\cdot,\cdot]\!] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ a Lie bracket on $\Gamma(A)$ such that

$$\{\hat{X}, \hat{Y}\}_{A^*} = -\widehat{[\![X, Y]\!]}, \ \ \forall X, Y \in \Gamma(A)$$

- $\exists \rho : A \to TQ$ a bundle map (the anchor map) such that

$$\{\hat{X}, f \circ \tau_{A^*}\}_{A^*} = -\rho(X)(f) \circ \tau_{A^*}, \ \ \forall f \in C^\infty(Q)$$

$$[\![X, fY]\!] = f[\![X, Y]\!] + \rho(X)(f)Y$$

In other words, $(A, [\![\cdot,\cdot]\!], \rho)$ is a Lie algebroid

$\Gamma(\Lambda^k(A^*))$ the space of "k-forms"

The differential d^A on A

$$\alpha \in \Gamma(\Lambda^k(A^*)) \Longrightarrow d^A\alpha \in \Gamma(\Lambda^{k+1}(A^*))$$

$$(d^A\alpha)(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i \rho(X_i)(\alpha(X_0, \dots, \widehat{X}_i, \dots, X_k))$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha([\![X_i, X_j]\!], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k),$$

A natural construction to generate left infinitesimal Poisson actions on A^* with equivariant momentum map

We only need a Lie algebra anti-morphism from a real Lie algebra $\mathfrak{g}$ on $\Gamma(A)$

$$\psi : \mathfrak{g} \to \Gamma(A)$$

$$\Downarrow$$

- The left infinitesimal Poisson action of $\mathfrak{g}$ on A^*

$$\psi_{A^*} : \mathfrak{g} \to \mathfrak{X}(A^*), \quad \psi_{A^*}(\xi) = X_{\widehat{\psi(\xi)}}$$

  $X_{\widehat{\psi(\xi)}}$ is the hamiltonian vector field of $\widehat{\psi(\xi)}$
- The equivariant momentum map

$$J : A^* \to \mathfrak{g}^* \quad J(\alpha)(\xi) = \alpha(\psi(\xi)), \quad \text{for } \alpha \in A^* \text{ and } \xi \in \mathfrak{g}$$

Remark

$X_{\widehat{\psi(\xi)}}$ is τ_{A^*}-projectable over the vector field $\rho(\psi(\xi))$ on M

$$\Downarrow$$

A left infinitesimal action $\psi_M : \mathfrak{g} \to \mathfrak{X}(M)$ of $\mathfrak{g}$ on M which is covered by ψ_{A^*}

We want to obtain actions of Lie groups on linear Poisson manifolds with equivariant momentum map

A solution: To integrate the previous infinitesimal actions

What is the result?

An action of the Lie group G on A^* "by complete lifts"

A linear action of G on A^* by complete lifts with respect to $\psi : \mathfrak{g} \to \Gamma(A)$

A linear Poisson action Φ^* of G on A^*

$$\Phi^* : G \times A^* \to A^*, \;\; \Phi^*(g, \alpha) = \Phi^*_{g^{-1}}\alpha$$

with equivariant momentum map $J : A^* \to \mathfrak{g}^*$ defined by

$$J(\alpha)(\xi) = \alpha(\psi(\xi))$$

for $\alpha \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$

The linear action Φ^* covers an action ϕ of G on M

1. Actions by complete lifts on linear Poisson structures

- The dual linear action of G on A

$$\Phi : G \times A \to A, \quad \Phi(g, a) = \Phi_g a$$

- The left infinitesimal action

$$\psi_A : \mathfrak{g} \to \mathfrak{X}(A), \quad \xi \to \psi_A(\xi) = \psi(\xi)^c$$

$$\psi(\xi)^c \equiv \text{the complete lift of } \psi(\xi) \in \Gamma(A)$$

1. Actions by complete lifts on linear Poisson structures

The aim of this talk

- $\surd$ To show that an action of a Lie group G by complete lifts on a linear Poisson manifold A^* admits a natural equivariant momentum map
- To prove that the linear Poisson structure on A^* is covered by a symplectic Lie algebroid
- To prove that the reduction of the linear Poisson structure is (under certain regularity conditions) covered by a (reduced) symplectic Lie algebroid (for this purpose, a Marsden-Weinstein reduction theorem for a general symplectic Lie algebroid will be discussed)
- To show under what conditions the reduction of the standard symplectic Lie algebroid is again a standard symplectic Lie algebroid

It is a symplectic Lie algebroid over A^*

The A-tangent bundle to A^*; $\mathcal{T}^A A^*$

The fiber over $\gamma \in A_q^*$:

$$\mathcal{T}_\gamma^A A^* = \{(a, v_\gamma) \in A_q \times T_\gamma A^* / (T_\gamma \tau_{A*})(v_\gamma) = \rho(a)\}$$

$\mathcal{T}^A A^*$ is a vector bundle over A^* of rank $2n$

$X \in \Gamma(A)$ and $U \in \mathfrak{X}(A^*)$ / U is τ_{A^*}-projectable over $\rho(X)$

$$\Downarrow$$

(X, U) is a projectable section of $\mathcal{T}^A A^*$

Remark: $\Gamma(\mathcal{T}^A A^*)$ is locally generated by $2n$ projectable sections

2. The symplectic cover of a linear Poisson manifold

The Lie algebroid structure on $\mathcal{T}^A A^*$

- The Lie bracket on $\Gamma(\mathcal{T}^A A^*)$:

$$(X, U), (Y, V) \in \Gamma(A) \times \mathfrak{X}(A^*) \text{ projectable sections} \Longrightarrow$$

$$[\![(X, U), (Y, V)]\!]_{\mathcal{T}^A A^*} = ([\![X, Y]\!], [U, V])$$

- The anchor map:

$$\rho_{\mathcal{T}^A A^*}(a, v_\gamma) = v_\gamma, \quad (a, v_\gamma) \in \mathcal{T}^A_\gamma A^*$$

Remark: $A = TQ \Longrightarrow \mathcal{T}^A A^* \simeq T(T^*Q)$

The canonical symplectic section of $\mathcal{T}^A A^*$

- The Liouville section θ_A:

$$\theta_A(\gamma)(a, v_\gamma) = \gamma(v), \quad for(a, v_\gamma) \in \mathcal{T}^A_\gamma A^*$$

- The canonical symplectic section

$$\Omega_A \in \Gamma(\Lambda^2(\mathcal{T}^A A^*)^*); \;\; \Omega_A = -d^{\mathcal{T}^A A^*}\theta_A$$

($A = TQ \Longrightarrow \Omega_A$ is the canonical symplectic structure on T^*Q)

Ω_A is non-degenerate and closed (exact)

$(\mathcal{T}^A A^*, \Omega_A)$ is the standard symplectic Lie algebroid

2. The symplectic cover of a linear Poisson manifold

Hamiltonian mechanics:

$$H : A^* \to \mathbb{R} \in C^\infty(A^*) \text{ a hamiltonian function } \Longrightarrow$$

$$d^{\mathcal{T}^A A^*} H \in \Gamma((\mathcal{T}^A A^*)^*) \Longrightarrow \exists! \mathcal{H}_H \in \Gamma(\mathcal{T}^A A^*) / \iota_{\mathcal{H}_H} \Omega_A = d^{\mathcal{T}^A A^*} H$$

$$\mathcal{H}_H \equiv \text{the hamiltonian section of } H \text{ with respect to } \Omega_A$$

Theorem (M de León, JCM, E Martínez, 2005)

The symplectic structure of $\mathcal{T}^A A^*$ covers the linear Poisson structure of A^*, that is,

$$\{F, G\}_{A^*} = \Omega_A(\mathcal{H}_F, \mathcal{H}_G), \forall F, G \in C^\infty(A^*)$$

The aim of this talk

- $\surd$ To show that an action of a Lie group G by complete lifts on a linear Poisson manifold A^* admits a natural equivariant momentum map
- $\surd$ To prove that the linear Poisson structure on A^* is covered by a symplectic Lie algebroid
- To prove that the reduction of the linear Poisson structure is (under certain regularity conditions) covered by a (reduced) symplectic Lie algebroid (for this purpose, a Marsden-Weinstein reduction theorem for a general symplectic Lie algebroid will be discussed)
- To show under what conditions the reduction of the standard symplectic Lie algebroid is again a standard symplectic Lie algebroid

3. Marsden-Weinstein reduction of symplectic Lie algebroids and linear Poisson structures

$(E, [\![\cdot, \cdot]\!], \rho)$ a Lie algebroid over a manifold P

E is a symplectic Lie algebroid

$$\Updownarrow$$

$\exists \Omega \in \Gamma(\Lambda^2 E^*)$ which is closed and non-degenerate

closed $\Longleftrightarrow d^E \Omega = 0$

non-degenerate $\Longleftrightarrow$ the map $\flat_\Omega : \Gamma(E) \to \Gamma(E^*)$ defined by $\flat_\Omega(X) = \iota_X \Omega$ is an isomorphism of $C^\infty(P)$-modules

Examples: i) M a symplectic manifold $\Rightarrow TM$ a symplectic Lie algebroid
ii) A a Lie algebroid $\Rightarrow \mathcal{T}^A A^*$ is a symplectic Lie algebroid

3. Marsden-Weinstein reduction of symplectic Lie algebroids and linear Poisson structures

Hamiltonian dynamics:

$$H : P \to \mathbb{R} \in C^\infty(P) \text{ a hamiltonian function } \Longrightarrow$$

$$d^E H \in \Gamma(E^*) \Longrightarrow \exists! \mathcal{H}_H \in \Gamma(E) / \flat_\Omega(\mathcal{H}_H) = d^E H$$

$$\mathcal{H}_H \equiv \text{the hamiltonian section of } H \text{ with respect to } \Omega$$

Theorem (K Mackenzie, P Xu, 1995; Y Kosmann-Schwarzbach, 1995)

For $F, G \in C^\infty(P)$ define

$$\{F, G\}_P = \Omega(\mathcal{H}_F, \mathcal{H}_G).$$

Then, $\{\cdot,\cdot\}_P$ is a Poisson bracket on P and the hamiltonian vector field X_H of $H \in C^\infty(P)$ with respect to $\{\cdot,\cdot\}_P$ is

$$X_H = \rho(\mathcal{H}_H)$$

Hamiltonian action of a Lie group G on E

- An action (Φ, ϕ) of G on E by complete lifts with respect to a Lie algebra anti-morphism $\psi : \mathfrak{g} \to \Gamma(E)$. So, the infinitesimal generator of Φ (respectively, ϕ) associated with $\xi \in \mathfrak{g}$ is $\xi_E = \psi(\xi)^c$ (respectively, $\xi_Q = \rho(\psi(\xi))$)
- The action (Φ, ϕ) is symplectic, that is,

$$(\Phi_g, \phi_g)^* \Omega = \Omega, \ \ \forall g \in G$$

and, in addition, it admits an equivariant momentum map $J : P \to \mathfrak{g}^*$, that is,

$$\iota_{\psi(\xi)} \Omega = d^E \hat{J}_\xi, \ \ \forall \xi \in \mathfrak{g}$$

Remark: If $\Psi \in \Gamma(\Lambda^2 E^*)$ then $(\Phi_g, \phi_g)^* \Psi \in \Gamma(\Lambda^2 E^*)$ is given by

$$((\Phi_g, \phi_g)^* \Psi)(x)(e, e') = \Psi(\phi_g(x))(\Phi_g(e), \Phi_g(e')), \ \ \forall e, e' \in E_x$$

A consequence:
(ϕ, J) is a hamiltonian action of G on P, that is, ϕ is a Poisson action of G on P and $J : P \to \mathfrak{g}^*$ is an equivariant momentum map for this action. So, we can apply Marsden-Ratiu theorem

Corollary

- If $\mu \in \mathfrak{g}^*$ is a regular value of J then the isotropy group G_μ acts on the submanifold $J^{-1}(\mu)$
- If the action of G_μ on $J^{-1}(\mu)$ is free and proper then the space of orbits $P_\mu = J^{-1}(\mu)/G_\mu$ admits a Poisson structure $\{\cdot,\cdot\}_\mu$ which is characterized by the condition
$$\{\tilde{f}, \tilde{g}\}_\mu \circ \pi_\mu = \{f, g\} \circ \iota_\mu$$
for $\tilde{f}, \tilde{g} \in C^\infty(P_\mu)$, where $\pi_\mu : J^{-1}(\mu) \to P_\mu$ is the canonical projection, $\iota_\mu : J^{-1}(\mu) \to P$ is the canonical inclusion and $f, g \in C^\infty(P)$ are G-invariant extensions of $\tilde{f} \circ \pi_\mu$ and $\tilde{g} \circ \pi_\mu$, respectively.

A natural question arise: Is covered the (reduced) Poisson structure on P_μ by a (reduced) symplectic Lie algebroid?

The answer: Yes (under certain regularity condition)

Marsden-Weinstein reduction theorem for symplectic Lie algebroids

If $T_xJ \circ \rho : E_x \to T_\mu \mathfrak{g}^*$ is surjective, for all $x \in J^{-1}(\mu)$, then there exists an affine action of the Lie group TG_μ on the Lie subalgebroid $(J^T)^{-1}(0,\mu)$ such that the space of orbits $E_\mu = (J^T)^{-1}(0,\mu)/TG_\mu$ is a symplectic Lie algebroid over $P_\mu = J^{-1}(\mu)/G_\mu$ which covers the Poisson structure $\{\cdot,\cdot\}_\mu$ on P_μ. Moreover, the symplectic section Ω_μ of E_μ is characterized by the condition

$$\widetilde{\pi}_\mu^*(\Omega_\mu) = \widetilde{\iota}_\mu^*(\Omega),$$

where $\widetilde{\pi}_\mu : (J^T)^{-1}(0,\mu) \to A_\mu$ is the canonical projection and $\widetilde{\iota}_\mu : (J^T)^{-1}(0,\mu) \to A$ is the canonical inclusion.

JCM, E Padrón, M Rodríguez-Olmos (Preprint, 2011)

3. Marsden-Weinstein reduction of symplectic Lie algebroids and linear Poisson structures

- The Lie algebroid morphism $J^T : E \to T\mathfrak{g}^* \simeq \mathfrak{g}^* \times \mathfrak{g}^*$

$$J^T(e) = ((dJ)(\rho(e)), J(\tau_E(e)))$$

- The affine action of $TG_\mu \simeq G_\mu \times \mathfrak{g}_\mu$ on $(J^T)^{-1}(0, \mu)$

$$(g, \xi) \cdot e = \Phi_g(e) + \Phi_g(\psi(\xi)(\tau_E(e)))$$

3. Marsden-Weinstein reduction of symplectic Lie algebroids and linear Poisson structures

The aim of this talk

- $\surd$ To show that an action of a Lie group G by complete lifts on a linear Poisson manifold A^* admits a natural equivariant momentum map
- $\surd$ To prove that the linear Poisson structure on A^* is covered by a symplectic Lie algebroid
- $\surd$ To prove that the reduction of the linear Poisson structure is (under certain regularity conditions) covered by a (reduced) symplectic Lie algebroid (for this purpose, a Marsden-Weinstein reduction theorem for a general symplectic Lie algebroid will be discussed)
- To show under what conditions the reduction of the standard symplectic Lie algebroid is again a standard symplectic Lie algebroid

4. Marsden-Weinstein reduction of the standard symplectic Lie algebroid

Next step: To apply our reduction theorem to the particular case when the symplectic Lie algebroid E is $\mathcal{T}^A A^*$

We will need a hamiltonian action of G on $\mathcal{T}^A A^*$ from an action of G on A by complete lifts

(Φ, ϕ) an action of G on A by complete lifts with associated Lie algebra anti-morphism $\psi : \mathfrak{g} \to \Gamma(A)$

$$\Downarrow$$

A hamiltonian action $((\Phi, T\Phi^*), (\psi, \psi_{A^*}))$ of G on the symplectic Lie algebroid $\mathcal{T}^A A^*$

4. Marsden-Weinstein reduction of the standard symplectic Lie algebroid

The couple $((\Phi, T\Phi^*), (\psi, \psi_{A^*}))$

- The action $(\Phi, T\Phi^*) : G \times \mathcal{T}^A A^* \to \mathcal{T}^A A^*$

$$(\Phi, T\Phi^*)(g, (a, v_\gamma)) = (\Phi_g(a), (T_\gamma \Phi^*_{g^{-1}})(v_\gamma))$$

- The Lie algebra anti-morphism $(\psi, \psi_{A^*}) : \mathfrak{g} \to \Gamma(\mathcal{T}^A A^*)$

$$(\psi, \psi_{A^*})(\xi) = (\psi(\xi), \psi_{A^*}(\xi) = X_{\widehat{\psi(\xi)}})$$

Now, we can apply our general reduction theorem to the symplectic Lie algebroid $\mathcal{T}^A A^*$

4. Marsden-Weinstein reduction of the standard symplectic Lie algebroid

The equivariant momentum map

$$J_{A^*} : A^* \to \mathfrak{g}^*$$

$$J_{A^*}(\alpha_x)(\xi) = \alpha_x(\psi(\xi)(x)), \;\; \alpha_x \in A^*_x, \;\; \xi \in \mathfrak{g}$$

The tangent equivariant map

$$J^T_{A^*} : \mathcal{T}^A A^* \to T\mathfrak{g}^* \simeq \mathfrak{g}^* \times \mathfrak{g}^*$$

$$J^T_{A^*}(a_x, v_{\alpha_x}) = ((dJ_{A^*})(v_{\alpha_x}), J_{A^*}(\alpha_x)), \; (a_x, v_{\alpha_x}) \in \mathcal{T}^A_{\alpha_x} A^*$$

The reduced symplectic Lie algebroid (for a free and proper action of G on M)

$$(\mathcal{T}^A A^*)_\mu = \frac{(J^T_{A^*})^{-1}(0, \mu)}{TG_\mu} \to \frac{J^{-1}_{A^*}(\mu)}{G_\mu}$$

4. Marsden-Weinstein reduction of the standard symplectic Lie algebroid

A new problem arise:

Does there exists a (reduced) Lie algebroid $\bar{A}$ such that the Marsden-Weinstein reduction of the symplectic Lie algebroid $\mathcal{T}^A A^*$ is symplectomorphic to $\mathcal{T}^{\bar{A}} \bar{A}^*$?

JCM, E Padrón, M Rodríguez-Olmos (Preprint, 2011)

It is the symplectic Lie algebroid counterpart of the cotangent bundle reduction theory (Abraham-Marsden, Kummer, Guichardet, Iwai, Montgomery,...)

4. Marsden-Weinstein reduction of the standard symplectic Lie algebroid

A particular case: $\mu = 0$ ($\Rightarrow G_\mu = G$)

The Lie algebroid A_0

The reduced Poisson manifold $J_{A^*}^{-1}(0)/G$ is a vector bundle over Q/G and the Poisson structure on $J_{A^*}^{-1}(0)/G$ is linear. So, the dual bundle $A_0 = (J_{A^*}^{-1}(0)/G)^*$ is a Lie algebroid over Q/G.

The reduced symplectic Lie algebroid (the last result in the talk by E Martínez when the reductive ideal is the set of tangent vectors to the orbits of the G-action)

The reduced symplectic Lie algebroid

$$(\mathcal{T}^A A^*)_0 = (J_{A^*}^T)^{-1}(0,0)/TG \to J_{A^*}^{-1}(0)/G$$

is symplectomorphic to the standard symplectic Lie algebroid $\mathcal{T}^{A_0} A_0^* \to A_0^*$.

4. Marsden-Weinstein reduction of the standard symplectic Lie algebroid

The general case: $\mu \in \mathfrak{g}^*$ an arbitrary regular value
$i : \mathfrak{g}_\mu \to \mathfrak{g}$ the inclusion; $i^* : \mathfrak{g}^* \to \mathfrak{g}_\mu^*$ the projection

$$J^\mu_{A^*} = i^* \circ J_{A^*} : A^* \to \mathfrak{g}^*_\mu$$

The Lie algebroid A_0^μ

The quotient space $(A_0^\mu)^* = (J^\mu_{A^*})^{-1}(0)/G_\mu$ is a vector bundle over Q/G_μ which admits a linear Poisson structure. So, the dual bundle $A_0^\mu = ((J^\mu_{A^*})^{-1}(0)/G_\mu)^*$ is a Lie algebroid over Q/G_μ.

4. Marsden-Weinstein reduction of the standard symplectic Lie algebroid

The reduced symplectic Lie algebroid

There exists a symplectic Lie algebroid monomorphism $\mathcal{I}$ between the reduced symplectic Lie algebroid

$$(\mathcal{T}^A A^*)_\mu = (J^T_{A^*})^{-1}(0,\mu)/TG_\mu \to J^{-1}_{A^*}(\mu)/G_\mu$$

and the symplectic Lie algebroid $(\mathcal{T}^{A_0^\mu}(A_0^\mu)^*, \Omega_{A_0^\mu} - (pr_1)^*(B_\mu))$, where $pr_1 : \mathcal{T}^{A_0^\mu}(A_0^\mu)^* \to A_0^\mu$ is the canonical projection on the first factor, $B_\mu \in \Gamma(\Lambda^2(A_0^\mu)^*)$ and $d^{A^0_\mu} B_\mu = 0$. In the particular case when $G_\mu = G$ then $\mathcal{I}$ is a isomorphism.

Remark: B_μ is the "magnetic term". It is obtained from a principal connection on the principal G_μ-bundle $\pi_\mu : Q \to Q/G_\mu$.

4. Marsden-Weinstein reduction of the standard symplectic Lie algebroid

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5. Future work

- To find a more suitable definition of a momentum map for a Poisson action

- To develop hamiltonian reduction by stages for Poisson structures which are covered by symplectic Lie algebroids (following the ideas contained in the recent book by Marsden, Misiolek, Ortega, Perlmutter and Ratiu, 2007, for the particular case of symplectic structures).

- To discuss singular reduction of Poisson structures which are covered by symplectic Lie algebroids (references for singular reduction of symplectic structures are contained in the previous book)

THANKS!