Hamiltonian systems on Lie algebroids:

Variational description and momentum conservation

Poisson Geometry and Applications

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Abstract

- Give a variational formalism for Hamiltonian systems on Lie algebroids.
- Describe conservation of momentum in terms of connections.
- Explore reduction theory in this setting.

Work in progress.

Several reasons for formulating Mechanics on Lie algebroids

- The inclusive nature of the Lie algebroid framework: under the same formalism one can consider standard mechanical systems, systems on Lie algebras, systems on semidirect products, systems with symmetries.
- The reduction of a mechanical system on a Lie algebroid is a mechanical system on a Lie algebroid, and this reduction procedure is done via morphisms of Lie algebroids.
- Well adapted: the geometry of the underlying Lie algebroid determines some dynamical properties as well as the geometric structures associated to it (e.g. Symplectic structure). Provides a natural way to use quasi-velocities in Mechanics.

Lie Algebroids

A Lie algebroid structure on a vector bundle $\tau\colon E \to M$ is given by

- a Lie algebra structure $(\mathrm{Sec}(E), [\,,\,])$ on the set of sections of E,

$$\sigma, \eta \in \mathrm{Sec}(E) \qquad \Rightarrow \qquad [\sigma, \eta] \in \mathrm{Sec}(E)$$

- a morphism of vector bundles $\rho\colon E \to TM$ over the identity, such that

$$[\sigma, f\eta] = f[\sigma, \eta] + (\rho(\sigma)f)\,\eta,$$

  where $\rho(\sigma)(m) = \rho(\sigma(m))$. The map ρ is said to be **the anchor**.

As a consequence of the Jacobi identity

$$\rho([\sigma, \eta]) = [\rho(\sigma), \rho(\eta)]$$

Examples

- **Tangent bundle.**

  $E = TM,$

  $\rho = \mathrm{id},$

  $[\,,\,] =$ bracket of vector fields.

- **Tangent bundle and parameters.**

  $E = TM \times \Lambda \to M \times \Lambda,$

  $\rho\colon TM \times \Lambda \to TM \times T\Lambda, \qquad \rho\colon (v, \lambda) \mapsto (v, 0_\lambda),$

  $[\,,\,] =$ bracket of vector fields (with parameters).

Integrable subbundle.

$E \subset TM$, integrable distribution

$\rho = i$, canonical inclusion

$[\,,\,] =$ restriction of the bracket to vector fields in E.

Lie algebra.

$E = \mathfrak{g} \to M = \{e\}$, Lie algebra (fiber bundle over a point)

$\rho = 0$, trivial map (since $TM = \{0_e\}$)

$[\,,\,] =$ the bracket in the Lie algebra.

Atiyah algebroid.

Let $\pi\colon Q \to M$ a principal G-bundle.

$E = TQ/G \to M = Q/G$, (Sections are equivariant vector fields)

$\rho([v]) = T\pi(v)$ induced projection map

$[\,,\,] =$ bracket of equivariant vector fields (is equivariant).

Transformation Lie algebroid.

Let $\Phi\colon \mathfrak{g} \to \mathfrak{X}(M)$ be an action of a Lie algebra $\mathfrak{g}$ on M.

$E = M \times \mathfrak{g} \to M$,

$\rho(m, \xi) = \Phi(\xi)(m)$ value of the fundamental vector field

$[\,,\,] =$ induced by the bracket on $\mathfrak{g}$.

Lie algebroid associated to a Poisson structure.

Let (M, π) a Poisson manifold;

$\pi\colon T^*M \times T^*M \to \mathbb{R}$

$\{f, g\} = \pi(df, dg)$

$E = T^*M \to M$

$\rho\colon T^*M \to TM$; $\rho(\alpha) = \pi(\ ,\alpha)$

$[\,,\,]$ is the Koszul bracket

$$[\alpha, \beta] = \mathcal{L}_{\rho(\alpha)}\beta - \mathcal{L}_{\rho(\beta)}\alpha - d\big(\pi(\alpha, \beta)\big).$$

They are determined by the relations

$$\rho(df) = X_f,$$

$$[df, dg] = d\{f, g\}$$

Structure functions

A local coordinate system (x^i) in the base manifold M and a local basis of sections (e_α) of E, determine a local coordinate system (x^i, y^α) on E,

$$y^\alpha(a) = \langle\, e^\alpha\,, a\,\rangle.$$

The anchor and the bracket are locally determined by the local functions $\rho^i_\alpha(x)$ and $C^\alpha_{\beta\gamma}(x)$ on M given by

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial x^i}$$

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}\; e_\gamma,$$

called the **structure functions**.

The functions ρ^i_α and $C^\alpha_{\beta\gamma}$ satisfy some relations due to the compatibility condition and the Jacobi identity which are called the **structure equations**:

- $[\rho(e_\alpha), \rho(e_\beta)] = \rho([e_\alpha, e_\beta])$

$$\boxed{\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma C^\gamma_{\alpha\beta}}$$

- $[e_\alpha, [e_\beta, e_\gamma]] + [e_\beta, [e_\gamma, e_\alpha]] + [e_\gamma, [e_\alpha, e_\beta]] = 0$

$$\boxed{\sum_{\text{cyclic}(\alpha,\beta,\gamma)} \left[\rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial x^i} + C^\mu_{\beta\gamma} C^\nu_{\alpha\mu}\right] = 0.}$$

Exterior calculus

Let of $\tau \colon E \to M$ be a Lie algebroid and consider the exterior algebra $\bigwedge E^* \to M$ of its dual.

If we think of a Lie algebroid as an alternative tangent bundle, we should think of a section of $\bigwedge E^*$ as an alternative differential form.

Sections of $\bigwedge^p E^*$ are called E-differential forms or just E-forms. As usual, a 0-form is just a function on the base.

A Lie algebroid structure on E is equivalent to the existence of a differential operator d taking k-forms into $(k+1)$-forms and satisfying

- d is a graded derivation of degree 1

$$d(\theta \wedge \omega) = d\theta \wedge \omega + (-1)^{\mathrm{degree}(\theta)} \theta \wedge d\omega.$$

- $d \circ d = 0$.

Exterior differential

On 0-forms

$$df(\sigma) = \rho(\sigma)f$$

On p-forms ($p > 0$)

$$\begin{aligned} d\omega(\sigma_1, \ldots, \sigma_{p+1}) = \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\sigma_i)\omega(\sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \sigma_{p+1}) \\ &- \sum_{i<j} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_1, \ldots, \widehat{\sigma_i}, \ldots, \widehat{\sigma_j}, \ldots, \sigma_{p+1}). \end{aligned}$$

Poisson bracket

The dual E^* of a Lie algebroid carries a canonical Poisson structure. In terms of linear and basic functions, the Poisson bracket is defined by

$$\{\hat{\sigma}, \hat{\eta}\} = \widehat{[\sigma, \eta]}$$
$$\{\hat{\sigma}, \tilde{g}\} = \rho(\sigma) g$$
$$\{\tilde{f}, \tilde{g}\} = 0$$

for f, g functions on M and σ, η sections of E.

Basic and linear functions are defined by

$$\begin{aligned} \tilde{f}(\mu) &= f(m) \\ \hat{\sigma}(\mu) &= \langle\, \mu\,, \sigma(m)\,\rangle \end{aligned} \qquad \text{for } \mu \in E^*_m.$$

In coordinates

$$\{x^i, x^j\} = 0 \qquad \{\mu_\alpha, x^j\} = \rho^i_\alpha \qquad \{\mu_\alpha, \mu_\beta\} = C^\gamma_{\alpha\beta}\mu_\gamma.$$

Mechanics on Lie algebroids

Lie algebroid $E \to M$.

$L \in C^\infty(E)$ or $H \in C^\infty(E^*)$

- $E = TM \to M$ Standard classical Mechanics
- $E = \mathcal{D} \subset TM \to M$ (integrable) System with holonomic constraints
- $E = TQ/G \to M = Q/G$ System with symmetry (eg. Classical particle on a Yang-Mills field)
- $E = \mathfrak{g} \to \{e\}$ System on a Lie algebra (eg. Rigid body)
- $E = M \times \mathfrak{g} \to M$ System on a semidirect product (eg. Heavy top)

Hamilton equations

E^* is a poisson manifold, so that, given $H \in C^\infty(E^*)$ we have a dynamical system

$$\dot{F} = \{F, H\}.$$

In coordinates, **Hamilton equations** are

$$\begin{aligned}
\frac{dx^i}{dt} &= \rho^i_\alpha \frac{\partial H}{\partial \mu_\alpha} \\
\frac{d\mu_\alpha}{dt} &= -\left(\mu_\gamma C^\gamma_{\alpha\beta} \frac{\partial H}{\partial \mu_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i} \right).
\end{aligned}$$

Lagrange's equations

(Weinstein 1996)

Given a function $L \in C^\infty(E)$, we define a dynamical system on E by means of a system of differential equations, which in local coordinates reads

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta = \rho^i_\alpha \frac{\partial L}{\partial x^i}$$
$$\dot{x}^i = \rho^i_\alpha y^\alpha.$$

The equation $\dot{x}^i = \rho^i_\alpha y^\alpha$ is the local expression of the admissibility condition: A curve $a\colon \mathbb{R} \to E$ is said to be **admissible** or an E-**path** if

$$\rho \circ a = \frac{d}{dt}(\tau \circ a).$$

Variational description

Formal variational description

Consider the action functional

$$\mathcal{S}(a) = \int_{t_0}^{t_1} L(a(t))\, dt$$

defined on curves on E with fixed base endpoints, which are moreover constrained to be E-paths.

But we also have to constraint the variations to be of the form

$$\delta x^i = \rho^i_\alpha \sigma^\alpha \qquad \delta y^\alpha = \dot{\sigma}^\alpha + C^\alpha_{\beta\gamma} a^\beta \sigma^\gamma$$

for some curve $\sigma(t)$ such that $\tau(a(t)) = \tau(\sigma(t))$ and $\sigma(t_0) = \sigma(t_1) = 0$.

Variation vector fields are of the form

$$\Xi_a(\sigma) = \rho^i_\alpha \sigma^\alpha \frac{\partial}{\partial x^i} + [\dot{\sigma}^\alpha + C^\alpha_{\beta\gamma} a^\beta \sigma^\gamma] \frac{\partial}{\partial y^\alpha}.$$

E-Homotopy

(Crainic and Fernandes 2003)

Let $I = [0,1]$ and $J = [t_0, t_1]$, and (s,t) coordinates in $\mathbb{R}^2$.

Two E-paths a_0 and a_1 are said to be **E-homotopic** if there exists a morphism of Lie algebroids $\Phi\colon TI \times TJ \to E$ such that

$$\Phi\left(\left.\frac{\partial}{\partial t}\right|_{(0,t)}\right) = a_0(t) \qquad \Phi\left(\left.\frac{\partial}{\partial s}\right|_{(s,t_0)}\right) = 0$$

$$\Phi\left(\left.\frac{\partial}{\partial t}\right|_{(1,t)}\right) = a_1(t) \qquad \Phi\left(\left.\frac{\partial}{\partial s}\right|_{(s,t_1)}\right) = 0.$$

In other words

$$\Phi = a_s(t)dt + b_s(t)ds$$

with $b_s(t_0) = 0$ and $b_s(t_1) = 0$. The curves a_s are the deformation of a_0, and the 'vector' b_s controls the variation. We have that

$$\frac{d}{ds}a_s(t)\Big|_{s=0} = \Xi_a(\sigma)$$

Homotopy foliation

The set of E-paths

$$\mathcal{A}(J,E) = \left\{ a\colon J \to E \;\middle|\; \rho \circ a = \frac{d}{dt}(\tau \circ a) \right\}$$

is a Banach submanifold of the Banach manifold of C^1-paths whose base path is C^2. Every E-homotopy class is a smooth Banach manifold and the partition into equivalence classes is a smooth foliation. The distribution tangent to that foliation is given by $a \in \mathcal{A}(J,E) \mapsto F_a$ where

$$F_a = \{\, \Xi_a(\sigma) \in T_a\mathcal{A}(J,E) \mid \sigma(t_0) = 0 \quad \text{and} \quad \sigma(t_1) = 0 \,\}.$$

and the codimension of F is equal to $\dim(E)$. The E-homotopy equivalence relation is regular if and only if the Lie algebroid is integrable (i.e. it is the Lie algebroid of a Lie groupoid).

Variational description

The E-**path space** with the appropriate differential structure is

$$\boxed{\mathcal{P}(J,E) = \mathcal{A}(J,E)_F}.$$

Fix $m_0, m_1 \in M$ and consider the set of E-paths with such base endpoints

$$\mathcal{P}(J,E)_{m_0}^{m_1} = \{\, a \in \mathcal{P}(J,E) \mid \tau(a(t_0)) = m_0 \quad \text{and} \quad \tau(a(t_1)) = m_1 \,\}\,.$$

It is a Banach submanifold of $\mathcal{P}(J,E)$.

Theorem

Let $L \in C^\infty(E)$ be a Lagrangian function on the Lie algebroid E and fix two points $m_0, m_1 \in M$. Consider the action functional $S\colon \mathcal{P}(J,E) \to \mathbb{R}$ given by $S(a) = \int_{t_0}^{t_1} L(a(t))dt$. The critical points of S on the Banach manifold $\mathcal{P}(J,E)_{m_0}^{m_1}$ are precisely those elements of that space which satisfy Lagrange's equations.

Morphisms and reduction

Given a morphism of Lie algebroids $\Phi\colon E \to E'$ the induced map $\hat{\Phi}\colon \mathcal{P}(J,E) \to \mathcal{P}(J,E')$ given by $\hat{\Phi}(a) = \Phi \circ a$ is smooth and $T\hat{\Phi}(\Xi_a(\sigma)) = \Xi_{\Phi\circ a}(\Phi \circ \sigma)$.

- If Φ is fiberwise surjective then $\hat{\Phi}$ is a submersion.
- If Φ is fiberwise injective then $\hat{\Phi}$ is a immersion.

Consider two Lagrangians $L \in C^\infty(E)$, $L' \in C^\infty(E')$ and $\Phi\colon E \to E'$ a morphism of Lie algebroids such that $L' \circ \Phi = L$.

Then, the action functionals S on $\mathcal{P}(J,E)$ and S' on $\mathcal{P}(J,E')$ are related by $\hat{\Phi}$, that is

$$S' \circ \hat{\Phi} = S.$$

Reduction

Theorem

Let $\Phi\colon E \to E'$ be a fiberwise surjective morphism of Lie algebroids. Consider a Lagrangian L on E and a Lagrangian L' on E' such that $L = L' \circ \Phi$. If a is a solution of Lagrange's equations for L then $a' = \Phi \circ a$ is a solution of Lagrange's equations for L'.

Proof

From $S' \circ \hat{\Phi} = S$ we get

$$\langle\, dS(a)\,,v\,\rangle = \langle\, dS'(\hat{\Phi}(a))\,,T_a\hat{\Phi}(v)\,\rangle = \langle\, dS'(a')\,,T_a\hat{\Phi}(v)\,\rangle.$$

Since $T_a\Phi(v)$ surjective, if $dS(a) = 0$ then $dS'(a') = 0$.

Reconstruction

Theorem

Let $\Phi\colon E \to E'$ be a morphism of Lie algebroids. Consider a Lagrangian L on E and a Lagrangian L' on E' such that $L = L' \circ \Phi$. If a is an E-path and $a' = \Phi \circ a$ is a solution of Lagrange's equations for L' then a itself is a solution of Lagrange's equations for L.

Proof

We have

$$\langle\, dS(a)\,,v\,\rangle = \langle\, dS'(a')\,,T_a\hat{\Phi}(v)\,\rangle.$$

If $dS'(a') = 0$ then $dS(a) = 0$.

Reduction by stages

Theorem

Let $\Phi_1 \colon E \to E'$ and $\Phi_2 \colon E' \to E''$ be fiberwise surjective morphisms of Lie algebroids. Let L, L' and L'' be Lagrangian functions on E, E' and E'', respectively, such that $L' \circ \Phi_1 = L$ and $L'' \circ \Phi_2 = L'$. Then the result of reducing first by Φ_1 and later by Φ_2 coincides with the reduction by $\Phi = \Phi_2 \circ \Phi_1$.

Examples.

Group actions.

G Lie group acting free and properly on a manifold Q, so that the quotient map $\pi: Q \to M$ is a principal bundle.

$E = TQ$ the standard Lie algebroid
$E' = TQ/G \to M$ Atiyah algebroid
$\Phi\colon E \to E'$, $\Phi(v) = [v]$ the quotient map

Φ is a fiberwise bijective Lie algebroid morphism.

Every G-invariant Lagrangian on TQ defines uniquely a Lagrangian L' on E' such that $L' \circ \Phi = L$.

Thus, the Euler-Lagrange equations on TQ reduce to the Lagrange-Poincaré equations on TQ/G.

Actions of Lie algebras (e.g. semidirect products).

Let G be a Lie group acting from the right on a manifold M.

$E = TG \times M \to G \times M$ where M is a parameter manifold
$E' = \mathfrak{g} \times M \to M$ transformation Lie algebroid
$\Phi(v_g, m) = (g^{-1}v_g, mg)$ is a fiberwise surjective morphism of Lie algebroids.

Consider a Lagrangian L on TG depending on the elements of M as parameters which is invariant by the joint action $L(g^{-1}\dot{g}, mg) = L(\dot{g}, m)$, and the reduced Lagrangian L' on E' by $L'(\xi, m) = L(\xi_G(e), m)$, so that $L' \circ \Phi = L$.

Euler-Lagrange equations on the group, with parameters, reduce to Euler-Poincaré equations with advected parameters.

Abelian Routh reduction.

A Lagrangian $L \in C^\infty(TQ)$ with cyclic coordinates θ and denote by q the other coordinates The Lagrangian L on TQ projects to a Lagrangian L' on TQ/G with the same coordinate expression. The solutions for L obviously project to solutions for L'.

The momentum $\mu = \frac{\partial L}{\partial \dot\theta}(q, \dot q, \dot\theta)$ is conserved and we can find $\dot\theta = \Theta(q, \dot q, \mu)$. The Routhian $R(q, \dot q, \mu) = L(q, \dot q, \Theta(q, \dot q, \mu) - \mu\dot\theta$ when restricted to a level set of the momentum $\mu = c$ defines a function L'' on $T(Q/G)$ which is just $L''(q, \dot q) = R(q, \dot q, c)$.

Thus $L''(q, \dot q) = L(q, \dot q, \Theta(q, \dot q, c)) - \frac{d}{dt}(c\theta)$, i.e. L and L'' differ on a total derivative. Lagrange equations reduce to $T(Q/G)$.

Hamilton's phase space variational principle

Hamilton's phase space principle: standard case

Standard case.

Consider curves $\mu(t) = (q^i(t), p_i(t)) \in T^*M$ and the functional

$$S_H(\mu) = \int_{t_0}^{t_1} \left[p_i(t)\dot{q}^i(t) - H(q^i(t), p_i(t)) \right] dt.$$

Solutions of Hamilton's differential equations are critical points of S_H on the set of curves with fixed base endpoints (but free values of p).

Alternatively, we can 'rephrase' the above as follows: look for critical points $(\mu(t), v(t)) \in T^*M \oplus TM$ of the functional

$$S(\mu, v) = \int_{t_0}^{t_1} \left[\langle\, \mu(t)\,, v(t)\,\rangle - H(\mu(t)) \right] dt,$$

with the restriction: $v(t) = \dot{q}(t) = \frac{d}{dt}\tau_M(v(t))$ (i.e. v is an admissible curve).

Hamilton's phase space principle: general case

Look for curves $(\mu(t), a(t)) \in E^* \oplus E$ which are critical points of the functional

$$S(\mu, a) = \int_{t_0}^{t_1} \big[\langle\, \mu(t)\,, a(t)\,\rangle - H(\mu(t))\big]\, dt,$$

where the curve a must be admissible with fixed base endpoints: $\tau(a(t_0)) = m_0$ and $\tau(a(t_1)) = m_1$.

Problem: $E^* \oplus E$ is not a Lie algebroid.

Solution: Take $(a(t), \dot{\mu}(t))$ instead of $(a(t), \mu(t))$. It takes value in the space

$$\mathcal{T}^E E^* = \{\, (a, v) \in E \times TE^* \mid T\pi(v) = \rho(a), \quad \text{and} \quad \tau(a) = \pi(\tau_{E^*}(v))\,\}$$

which is a Lie algebroid over E^*.

Admissible curves on $\mathcal{T}^E E^*$ are precisely those of the form $(a(t), \dot{\mu}(t))$ and two curves $(a, \dot{\mu})$, $(a', \dot{\mu}')$ are $\mathcal{T}^E E^*$-homotopic if and only if μ, μ' are homotopic (in the standard sense) and a, a' are E-homotopic.

Therefore we consider the manifold $\mathcal{P}(J, \mathcal{T}^E E^*)_{m_0}^{m_1}$. Vector tangent to it are of the form

$$\rho^i_\alpha \sigma^\alpha \frac{\partial}{\partial x^i} + \left(\dot\sigma^\alpha + C^\alpha_{\beta\gamma} a^\beta \sigma^\gamma\right) \frac{\partial}{\partial y^\alpha} + \zeta_\alpha \frac{\partial}{\partial \mu_\alpha} + \dot\zeta_\alpha \frac{\partial}{\partial \dot\mu_\alpha}.$$

with $\sigma(t_0) = 0$, $\sigma(t_1) = 0$.

In a more classical notation

$$\begin{aligned}
\delta x^i &= \rho^i_\alpha \sigma^\alpha \\
\delta y^\alpha &= \dot\sigma^\alpha + C^\alpha_{\beta\gamma} a^\beta \sigma^\gamma \\
\delta \mu_\alpha &= \zeta_\alpha
\end{aligned}$$

(and $\delta\dot\mu_\alpha = \dot\zeta_\alpha$, when needed).

Critical points of S on $\mathcal{P}(J, \mathcal{T}^E E^*)_{m_0}^{m_1}$:

$$\langle\, dS(\mu, a)\,, v \,\rangle = \int_{t_0}^{t_1} \left[a^\alpha \zeta_\alpha + \mu_\alpha \left(\dot\sigma^\alpha + C^\alpha_{\beta\gamma} a^\beta \sigma^\gamma\right) - \frac{\partial H}{\partial x^i} \rho^i_\alpha \sigma^\alpha - \frac{\partial H}{\partial \mu_\alpha} \zeta_\alpha \right] dt.$$

Integrating by parts and taking into account that $\sigma(t_0) = 0$, $\sigma(t_1) = 0$ we get

$$\langle\, dS(\mu, a)\,, v\,\rangle = \int_{t_0}^{t_1} \left[\left(a^\alpha - \frac{\partial H}{\partial \mu_\alpha} \right) \zeta_\alpha + \left(-\dot{\mu}_\alpha + \mu_\gamma C^\gamma_{\beta\alpha} a^\beta - \frac{\partial H}{\partial x^i} \rho^i_\alpha \right) \sigma^\alpha \right] dt.$$

Since σ and ζ are arbitrary

$$a^\alpha = \frac{\partial H}{\partial \mu_\alpha} \qquad \dot{\mu}_\alpha + \mu_\gamma C^\gamma_{\alpha\beta} a^\beta = -\rho^i_\alpha \frac{\partial H}{\partial x^i}.$$

We deduce that the curve $(\mu(t), a(t))$ is the solution of the differential equations

$$\boxed{\dot{x}^i = \rho^i_\alpha \frac{\partial H}{\partial \mu_\alpha} \qquad \dot{\mu}_\alpha + \mu_\gamma C^\gamma_{\alpha\beta} \frac{\partial H}{\partial \mu_\beta} = -\rho^i_\alpha \frac{\partial H}{\partial x^i}.}$$

Reduction and reconstruction

- Similar to the Lagrangian case.
- Can be easily generalized to optimal control theory.

Prolongation

Given a Lie algebroid $\tau\colon E \to M$ and a submersion $\nu\colon P \to M$ we construct the **E-tangent** to P (the prolongation of P with respect to E). It is the vector bundle $\tau_P^E\colon \mathcal{T}^E P \to P$ where the fiber over $p \in P$ is

$$\mathcal{T}_p^E P = \{\, (b, V) \in E_m \times T_p P \mid T\nu(V) = \rho(b) \,\}$$

Redundant notation: (p, b, V) for the element $(b, V) \in \mathcal{T}_p^E P$.

The bundle $\mathcal{T}^E P$ can be endowed with a structure of Lie algebroid.
The anchor $\rho^1\colon \mathcal{T}^E P \to TP$ is just the projection onto the third factor $\rho^1(p, b, V) = V$. The bracket is given in terms of projectable sections (σ, X), (η, Y)

$$[(\sigma, X), (\eta, Y)] = ([\sigma, \eta], [X, Y]).$$

The projection onto the second factor $\mathcal{T}\nu(p, b, V) = b$ is a morphism of Lie algebroids.

Local basis

Local coordinates (x^i, u^A) on P and a local basis $\{e_\alpha\}$ of sections of E, define a local basis $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$ of sections of $\mathcal{T}^E P$ by

$$\mathcal{X}_\alpha(p) = \left(p, e_\alpha(\pi(p)), \rho^i_\alpha \frac{\partial}{\partial x^i}\Big|_p\right) \qquad \text{and} \qquad \mathcal{V}_A(p) = \left(p, 0, \frac{\partial}{\partial u^A}\Big|_p\right).$$

The Lie brackets of the elements of the basis are

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = C^\gamma_{\alpha\beta}\, \mathcal{X}_\gamma, \qquad [\mathcal{X}_\alpha, \mathcal{V}_B] = 0 \qquad \text{and} \qquad [\mathcal{V}_A, \mathcal{V}_B] = 0,$$

and the exterior differential is determined by

$$\begin{aligned} dx^i &= \rho^i_\alpha \mathcal{X}^\alpha, & du^A &= \mathcal{V}^A, \\ d\mathcal{X}^\gamma &= -\frac{1}{2} C^\gamma_{\alpha\beta} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, & d\mathcal{V}^A &= 0, \end{aligned}$$

where $\{\mathcal{X}^\alpha, \mathcal{V}^A\}$ is the dual basis corresponding to $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$.

Prolongation of maps

If $\Psi\colon P \to P'$ is a bundle map over $\varphi\colon M \to M'$ and $\Phi\colon E \to E'$ is an admissible map over the same map φ then we can define a map $\mathcal{T}^{\Phi}\Psi\colon \mathcal{T}^{E}P \to \mathcal{T}^{E'}P'$ by means of

$$\mathcal{T}^{\Phi}\Psi(p, b, v) = (\Psi(p), \Phi(b), T_p\Psi(v)).$$

Theorem

Let Φ be an admissible map. Then, $\mathcal{T}^{\Phi}\Psi$ is a morphism of Lie algebroids if and only if Φ is a morphism of Lie algebroids.

The E-tangent to E

In particular, for $P = E$ we have the E-tangent to E

$$\mathcal{T}^E_a E = \{\, (b, v) \in E_m \times T_a E \mid T\tau(v) = \rho(b) \,\}\,.$$

The rank of this bundle is even: $\mathrm{Rank}(\mathcal{T}^E E) = 2\,\mathrm{Rank}(E)$ and has the following canonical structures:

- The **vertical endomorphism** $S\colon \mathcal{T}^E E \to \mathcal{T}^E E$

$$S(a, b, v) = (a, 0, b^{\mathsf{V}}_a),$$

- The **Liouville section** which is the vertical section corresponding to the Liouville dilation vector field:

$$\Delta(a) = (a, 0, a^{\mathsf{V}}_a).$$

Geometric Lagrangian Mechanics

Form the Lagrangian L we define the section θ_L of $(\mathcal{T}^E E)^*$, by $\theta_L = dL \circ S$, that is

$$\langle\, \theta_L\,, (a, b, V)\,\rangle = \frac{d}{ds} L(a + sb)\Big|_{s=0}.$$

Define the 2-form ω_L by

$$\omega_L = -d\theta_L\,,$$

and the energy

$$E_L = d_\Delta L - L.$$

and the Hamiltonian section Γ_L

$$i_\Gamma \omega_L = dE_L$$

When the Lagrangian is regular, ω_L is symplectic and the integral curves of the Hamiltonian vector field $\rho^1(\Gamma)$ are admissible curves.

Local expressions

- Liouville section and vertical endomorphism

$$\Delta = y^\alpha \mathcal{V}_\alpha \qquad \text{and} \qquad S = \mathcal{V}_\alpha \otimes \mathcal{X}^\alpha$$

- Cartan forms

$$\theta_L = \frac{\partial L}{\partial y^\alpha} \mathcal{X}^\alpha$$

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{X}^\alpha \wedge \mathcal{V}^\beta + \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^i \partial y^\alpha} \rho^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta} \rho^i_\alpha + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta,$$

- Energy

$$E_L = \frac{\partial L}{\partial y^\alpha} y^\alpha - L.$$

Symplectic Hamiltonian formalism

Consider the prolongation $\mathcal{T}^E E^*$ of the dual bundle $\pi\colon E^* \to M$:

$$\mathcal{T}^E E^* = \{\, (\mu, a, W) \in E^* \times E \times TE^* \mid \mu = \tau_{E^*}(W) \quad \rho(a) = T\pi(W) \,\}\,.$$

There is a **canonical symplectic structure** $\omega = -d\theta$, where the 1-form θ is defined by

$$\langle\, \theta_\mu \,, (\mu, a, W)\,\rangle = \langle\, \mu \,, a \,\rangle.$$

In coordinates

$$\theta = \mu_\alpha \mathcal{X}^\alpha,$$

and

$$\omega = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2}\mu_\gamma C^\gamma_{\alpha\beta}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta.$$

The Hamiltonian dynamics is given by the vector field $\rho(\Gamma_H)$ associated to the section Γ_H solution of the symplectic equation

$$i_{\Gamma_H}\omega = dH.$$

In coordinates, **Hamilton equations** are

$$\frac{dx^i}{dt} = \rho^i_\alpha \frac{\partial H}{\partial \mu_\alpha} \qquad\qquad \frac{d\mu_\alpha}{dt} = -\left(\mu_\gamma C^\gamma_{\alpha\beta}\frac{\partial H}{\partial \mu_\beta} + \rho^i_\alpha \frac{\partial H}{\partial x^i}\right).$$

The canonical Poisson bracket on E^* can be re-obtained by means of

$$\omega(dF, dG) = \{F, G\}$$

for $F, G \in C^\infty(E^*)$.

Momentum conservation

Momentum in local coordinates

Assume E regular, i.e. the rank of ρ is constant.

Take an adapted basis $\{e_a, e_A\}$ where $\{e_A\}$ is a basis of sections of $K = \mathrm{Ker}(\rho)$, and hence we have the dual basis $\{e^a, e^A\}$ and linear coordinates (x^i, μ_a, μ_A) for E^*.

The anchor is given by

$$\rho(e_b) = \rho^i_b \frac{\partial}{\partial x^i}, \qquad \rho(e_B) = 0,$$

and the brackets of the sections in the basis $\{e_a, e_A\}$ are

$$\begin{aligned} [e_A, e_B] &= C^D_{AB} e_D \\ [e_a, e_B] &= C^D_{aB} e_D \\ [e_a, e_b] &= C^c_{ab} e_c + C^D_{ab} e_D. \end{aligned} \tag{1}$$

In this class of coordinates, Hamilton differential equations are

$$\begin{aligned}
\frac{dx^i}{dt} &= \rho^i_a \frac{\partial H}{\partial \mu_a} \\
\frac{d\mu_a}{dt} &= -\rho^i_a \frac{\partial H}{\partial x^i} - \mu_\gamma C^\gamma_{a\beta} \frac{\partial H}{\partial \mu_\beta} \\
\frac{d\mu_A}{dt} &= -\mu_B C^B_{A\beta} \frac{\partial H}{\partial \mu_\beta}.
\end{aligned}$$

- **Question:** What is the intrinsic meaning of the last group of equations?

$$\frac{d\mu_A}{dt} = -\mu_B C^B_{A\beta} y^\beta \qquad \text{with } y^\beta = \frac{\partial H}{\partial \mu_\beta}.$$

- **Solution:** Evolution of momenta, obviously.

The momentum equation

General case: E Lie algebroid, not necessarily regular.

Consider an **ideal** of E: a subalgebroid $i\colon K \hookrightarrow E$

$$\sigma \in \mathrm{Sec}(E) \quad \text{and} \quad \eta \in \mathrm{Sec}(K) \quad \Rightarrow \quad [\xi, \eta] \in \mathrm{Sec}(K).$$

It follows that K is a bundle of Lie algebras, i.e., $\rho|_K = 0$.

Definition

Let $i\colon K \to E$ be the canonical inclusion of an ideal K into E. The **momentum map** J, with respect to K, is the dual map $J = i^*\colon E^* \to K^*$.

Explicitly, the momentum map $J\colon E^* \to K^*$ is the bundle map over the identity in M, given by

$$\langle\, J(\mu)\,,k\,\rangle = \langle\, \mu\,,k\,\rangle, \qquad \mu \in E^*, \quad k \in K.$$

The momentum equation

There exists a canonical linear E-connection on the vector bundle $K \to M$:

$$\nabla_\xi \eta = [\xi, \eta], \qquad \xi \in \mathrm{Sec}(E), \quad \eta \in \mathrm{Sec}(K).$$

It is flat connection (i.e. a representation):

$$\nabla_{\xi_1}\nabla_{\xi_2} - \nabla_{\xi_2}\nabla_{\xi_1} = \nabla_{[\xi_1,\xi_2]}.$$

Theorem

For any Hamiltonian function $H \in C^\infty(E^*)$, the momentum map satisfies the **momentum equation**:

$$\nabla_{\Gamma_H} J = 0.$$

Reduction

Since ∇ is a representation, it defines an orbit foliation of K^*. From $\nabla_{\Gamma_H} J = 0$, we have that the inverse image by J of an orbit is invariant under the dynamics.

In particular the zero section is one of such orbits,

$$J^{-1}(0) = \{\, \mu \in E^* \mid J(\mu) = 0 \,\} \simeq K^\circ.$$

The dual bundle to K° is canonically identified with the quotient vector bundle E/K, and this last inherits a Lie algebroid structure from E, because K is an ideal in E.

We consider the quotient Lie algebroid $F = E/K \to M$ and identify $J^{-1}(0) = F^*$.

We denote by $p\colon E \to F = E/K$ the quotient projection and by j its adjoint map $j = p^*\colon F^* \to E^*$, which is but the canonical inclussion of $J^{-1}(0)$ into E^*.

Canonical symplectic forms ω^E in $\mathcal{T}^E E^*$ and ω^F on $\mathcal{T}^F F^*$
Hamiltonian functions $H \in C^\infty(E^*)$ and $\bar{H} = H \circ j \in C^\infty(F^*)$,
Hamiltonian sections $\Gamma_H \in \mathrm{Sec}(\mathcal{T}^E E^*)$ and $\Gamma_{\bar{H}} \in \mathrm{Sec}(\mathcal{T}^F F^*)$.

To relate this objects we consider the maps $P = \mathcal{T}^p \mathrm{id} \colon \mathcal{T}^E F^* \to \mathcal{T}^F F^*$ and $I = \mathcal{T}^{\mathrm{id}} j \colon \mathcal{T}^E F^* \to \mathcal{T}^E E^*$ given by

$$P(\nu, a, w) = (\nu, p(a), w) \qquad \text{and} \qquad I(\nu, a, w) = (i(\nu), a, Tj(w)),$$

which are morphisms of Lie algebroids

$$\begin{tikzcd}
\mathcal{T}^E F^* \arrow[r, "I"] \arrow[d, "P"'] & \mathcal{T}^E E^* \\
\mathcal{T}^F F^* &
\end{tikzcd}$$

$$\begin{array}{ccc} \mathcal{T}^E F^* & \xrightarrow{I} & \mathcal{T}^E E^* \\ \Big\downarrow{\scriptstyle P} & & \\ \mathcal{T}^F F^* & & \end{array}$$

Theorem

- $P^\star \omega^F = I^\star \omega^E$.
- $P^\star(d\bar{H}) = I^\star(dH)$
- There exists $\tilde{\Gamma}_H$ section of $\mathcal{T}^E F$ such that $I \circ \tilde{\Gamma}_H = \Gamma_H \circ j$, and $\Gamma_{\bar{H}} = P \circ \tilde{\Gamma}_H$.

Summarizing

- We can describe Lagrangian and Hamiltonian systems on Lie algebroids by means of a variational formalism.
- Very appropriate for reduction.
- Momentum maps defined by ideals (but there are other alternatives).
- Momentum map is covariantly constant, with respect to a flat connection.
- Reduction at zero momentum value.

- To do: Reduction by stages.

Thank you!

Splittings

The quotient vector bundle has an induced Lie algebroid structure, so that we have

$$0 \longrightarrow K \xrightarrow{i} E \xrightarrow{p} F \longrightarrow 0$$

Take a spitting $\mathfrak{s}\colon F \to E$ and identify E with $F \oplus K$ by

$$(a, k) \simeq \mathfrak{s}(a) + i(k),$$

with the bracket

$$[(\sigma_1, \eta_1), (\sigma_2, \eta_2)]_{F\oplus K} = \Big([\sigma_1, \sigma_2]_F, [\eta_1, \eta_2]_K + \nabla_{\mathfrak{s}(\sigma_1)}\eta_2 - \nabla_{\mathfrak{s}(\sigma_2)}\eta_1 + \Omega(\sigma_1, \sigma_2)\Big),$$

Induces a splitting $\mathcal{T}^E E = \mathcal{T}^F F \times \mathcal{T}^K K$.

Canonical forms

Theorem

Let $\hat{\Omega}$ be the map whose value at the point $(\nu, \mu) \in F^* \oplus K^*$ is the bilinear form $\hat{\Omega}_{(\nu,\mu)} : \mathcal{T}^F_\nu F \times \mathcal{T}^F_\nu F \to \mathbb{R}$ given by

$$\hat{\Omega}_{(\nu,\mu)}\big((b_1, V_1), (b_2, V_2)\big) = \langle\, \mu \,, \Omega_\nu(b_1, b_2)\,\rangle.$$

Then we have

$$\omega^E(\varphi_1, \varphi_2) = \omega^F(\zeta_1, \zeta_2) + \omega^K(\gamma_1, \gamma_2) + \hat{\Omega}(\zeta_1, \zeta_2)$$

where φ_1, φ_2 are sections of $\mathcal{T}^E E^*$ such that $\Phi \circ \varphi_i = (\zeta_i, \gamma_i) \circ \phi$ for $i = 1, 2$.

It follows that the Hamilton equations are

$$i_{\Gamma_H^F}(\omega^F - \hat{\Omega}) = d_F H \qquad \text{and} \qquad i_{\Gamma_H^K}\omega^K = d_K H$$

where $\Phi \circ \Gamma_H = (\Gamma_H^F, \Gamma_H^K) \circ \phi$ and $d_F H$ and $d_K H$ are the components of dH, that is $\Phi^\star(d_F H, d_K H) = d^E H$.