Poisson brackets with prescribed Casimirs

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- Casimir functions
- Presentation of our problem
- Presentation of our results
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  - The operator $*$
  - Lepage's decomposition theorem
  - Relations between Ψ and $*$
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- Examples

Some standard notations

We denote by

- M a smooth, m-dimensional manifold ;
- TM and T^*M its tangent and its cotangent bundle ;
- $C^\infty(M)$ the space of smooth functions on M ;
- $\mathcal{V}^p(M)$, $p \in \mathbb{N}$, the space of smooth sections of $\bigwedge^p TM$, i.e., the space of smooth p-vector fields on M ;
- $\Omega^p(M)$, $p \in \mathbb{N}$, the space of smooth sections of $\bigwedge^p T^*M$, i.e., the space of smooth p-forms on M.

Poisson brackets

Poisson brackets, introduced by Siméon Denis Poisson on $\mathbb{R}^{2n}$ [9] and generalized to manifolds of arbitrary dimension by Sophus Lie [5], have an important role in *Hamiltonian dynamics, fluid dynamics, magnetohydrodynamics* and other fields of mathematical physics.

In modern language

A *Poisson bracket* on $C^\infty(M)$ is a bilinear map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

with the properties :

- $\{f,g\} = -\{g,f\}$;
- $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (Jacobi identity) ;
- $\{f,gh\} = \{f,g\}h + g\{f,h\}$ (biderivation - Leibniz's rule).

Thus, $(C^\infty(M), \{\cdot,\cdot\})$ has the structure of a Lie algebra.

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By virtue of the above properties, a Poisson bracket $\{\cdot,\cdot\}$ defines a bivector field Λ on M :

$$\Lambda(df, dg) = \{f, g\} \quad \text{and} \quad [\Lambda, \Lambda] = 0.$$

Reciprocally, any $\Lambda \in \mathcal{V}^2(M)$ that verifies $[\Lambda, \Lambda] = 0$, defines on M a Poisson bracket

$$\{f, g\} = \Lambda(df, dg).$$

A such Λ is called *Poisson tensor* and (M, Λ) *Poisson manifold*.

Classical examples

- **Symplectic manifolds**
  Any symplectic manifold (M, ω), ω is a nondegenerate closed smooth 2-form on M, is equipped with a Poisson bracket $\{\cdot, \cdot\}$ defined by ω as follows. Since $\omega^\flat : \mathcal{V}^1(M) \to \Omega^1(M)$, $X \mapsto \omega^\flat(X) = -\omega(X, \cdot)$ is an isomorphism, for any $f, g \in C^\infty(M)$,

$$\{f, g\} = \omega({\omega^\flat}^{-1}(df), {\omega^\flat}^{-1}(dg)).$$

- **The dual of a Lie algebra** $(\mathcal{G}, [\cdot, \cdot])$
  Let $M = \mathcal{G}^*$. For any $f, g \in C^\infty(M)$ and $x \in \mathcal{G}^*$, we define

$$\{f, g\}(x) := \langle x, [df(x), dg(x)] \rangle,$$

which is a linear Poisson bracket on $\mathcal{G}^*$.

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To a given Poisson tensor Λ on M, we can associate :

- A homomorphism $\Lambda^{\#} : \Omega^1(M) \to \mathcal{V}^1(M)$, $\alpha \mapsto \Lambda^{\#}(\alpha)$, such that, for any $\beta \in \Omega^1(M)$,

$$\langle \beta, \Lambda^{\#}(\alpha) \rangle = \Lambda(\alpha, \beta).$$

  - If $\alpha = df$, $f \in C^\infty(M)$, the vector field $X_f = \Lambda^{\#}(df) = \{f, \cdot\}$ is called *Hamiltonian vector field of f with respect to Λ.*
  - The image $\mathrm{Im}\Lambda^{\#} \subset \mathcal{V}^1(M)$ defines, as a completely integrable distribution on M, the *symplectic foliation of M* whose leaves are symplectic immersed submanifolds of (M, Λ).

- Its extension $\Lambda^{\#} : \Omega^p(M) \to \mathcal{V}^p(M)$, $p \in \mathbb{N}$, defined, for any $\zeta \in \Omega^p(M)$ and $\alpha_1, \ldots, \alpha_p \in \Omega^1(M)$, by

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Casimir functions

The elements of the center of Lie algebra $(C^\infty(M), \{\cdot,\cdot\})$, i.e., the functions $C \in C^\infty(M)$ such that

$$\{C, \cdot\} = 0 \Leftrightarrow \Lambda^\#(dC) = 0,$$

are called *Casimirs* of *Poisson structure* Λ.

They have a very important role in the analysis of Poisson structures because they are conserved quantities in any Hamiltonian system X_f of (M, Λ) :

$$X_f(C) = \langle dC, X_f\rangle = \{f, C\} = 0,$$

and, as such, they play a vital part in reducing the order of X_f and in its eventual integration.

Also, the symplectic leaves of (M, Λ) can be viewed as the common level sets of Casimirs of Λ.

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Presentation of our problem

To introduce our problem we remark that, for an arbitrary function C on $\mathbb{R}^3$, the bracket

$$\{x,y\} = \frac{\partial C}{\partial z}, \quad \{x,z\} = -\frac{\partial C}{\partial y} \quad \text{and} \quad \{y,z\} = \frac{\partial C}{\partial x}$$

(x, y, z being the coordinates functions of $\mathbb{R}^3$) is Poisson and it admits C as Casimir.

If $\Omega = dx \wedge dy \wedge dz$ is the standard volume element on $\mathbb{R}^3$, then the above bracket can be written as

$$\{x,y\}\Omega = dx \wedge dy \wedge dC \quad \Leftrightarrow \quad \{x,y\} = \frac{dx \wedge dy \wedge dC}{\Omega},$$

$$\{x,z\}\Omega = dx \wedge dz \wedge dC \quad \Leftrightarrow \quad \{x,z\} = \frac{dx \wedge dz \wedge dC}{\Omega},$$

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More generally, for given $C_1, C_2, \ldots, C_l$ functionally independent smooth functions on $\mathbb{R}^{l+2}$ and Ω a volume form on $\mathbb{R}^{l+2}$, the formula

$$\{g,h\}\Omega = dg \wedge dh \wedge dC_1 \wedge \ldots \wedge dC_l \Leftrightarrow$$
$$\{g,h\} = \frac{dg \wedge dh \wedge dC_1 \wedge \ldots \wedge dC_l}{\Omega}, \tag{1}$$

due to H. Flaschka and T. Ratiu, defines a Poisson bracket on $C^\infty(\mathbb{R}^{l+2})$ with $C_1, \ldots, C_l$ as Casimir invariants and with symplectic leaves of dimension at most 2.

J. Grabowski *et al*. have also proved [4] that : *Conversely, the bracket of any Poisson structure* Λ *on* $\mathbb{R}^{l+2}$ *with symplectic leaves of dimension at most* 2 *can be written as in (1).*

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The above bracket, called *Jacobian Poisson bracket* because

$$\{g, h\} = \det\left(\frac{\partial(g, h, C_1, \ldots, C_l)}{\partial(x_1, \ldots, x_{l+2})}\right),$$

firstly is used by P. Damianou [1, 2] for calculate the *transverse Poisson structures* to subregular nilpotent orbits of $\mathfrak{gl}(n, \mathbb{C})$ ($n \leq 7$). Later, this fact was extended by P. Damianou *et al.* [3] to *transverse Poisson structures* to subregular nilpotent orbits of any semi-simple Lie algebra $\mathcal{G}$.

Another interesting application of (1) is appeared in [6, 7, 8], where the polynomial Poisson algebras and their Heisenberg invariance properties are studied by V. Rubtsov *et al.*.

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In this lecture we are interested to extend formula (1) in the more general case of higher rank Poisson brackets.

Our problem can be formulated as follows :

For given $(m-2k)$ smooth functions $C_1, \ldots, C_{m-2k}$ on an m-dimensional smooth manifold M, functionally independent almost everywhere, and a volume element Ω on M, how we must write the $(m-2)$-form

$$\Phi = (\quad ? \quad) \wedge dC_1 \wedge \ldots \wedge dC_{m-2k}$$

such that the bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ defined by

$$\{h_1, h_2\}\Omega = dh_1 \wedge dh_2 \wedge \Phi \Leftrightarrow \{h_1, h_2\} = \frac{dh_1 \wedge dh_2 \wedge \Phi}{\Omega}$$

is a Poisson bracket of rank at most $2k$ with $C_1, \ldots, C_{m-2k}$ as Casimirs.

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We investigate this problem in the case where M is of even and of odd dimension, separately.

If $dimM = 2n$

We assume that M is endowed with a *suitable almost symplectic structure* ω_0 and we denote by Λ_0 the corresponding almost Poisson structure on M. Then, we prove that

$$\Phi = \Big(-\frac{1}{f}(\sigma + \frac{g}{k-1}\omega_0) \wedge \frac{\omega_0^{k-2}}{(k-2)!} \Big) \wedge dC_1 \wedge \ldots \wedge dC_{2n-2k},$$

where f satisfies $f^2 = \det\big(\{f_i, f_j\}_0\big) \neq 0$, σ is a 2-form on M satisfying certain special requirements and $g = i_{\Lambda_0}\sigma$.

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Presentation of our results

If $dimM = 2n+1$

We assume that M is equipped with a *suitable almost cosymplectic structure* (ϑ_0, Θ_0) and we denote by (Λ_0, E_0) the corresponding almost Jacobi structure on M. Then, we show that

$$\Phi = \Big(-\frac{1}{f}(\sigma + \frac{g}{k-1}\Theta_0) \wedge \frac{\Theta_0^{k-2}}{(k-2)!}\Big) \wedge dC_1 \wedge \ldots \wedge dC_{2n+1-2k},$$

where $f \in C^\infty(M)$ and $f \neq 0$ almost everywhere, σ is a 2-form on M satisfying certain particular conditions and $g = i_{\Lambda_0}\sigma$.

Tools for the demonstration

The key points in the establishment of the above formulas are

- the use of the operators Ψ and $*$;
- the relation which links Ψ and $*$;
- Lepage's decomposition theorem for differential forms.

The operator Ψ

Let M be a m-dimensional smooth manifold. We denote by

- $i_P : \Omega(M) \to \Omega(M)$ the *interior product of differential forms by a p-vector field P* defined, for any $\eta \in \Omega^q(M)$, $q \geq p$, and $Q \in \mathcal{V}^{q-p}(M)$, by
$$\langle i_P \eta, Q \rangle = (-1)^{(p-1)p/2} \langle \eta, P \wedge Q \rangle;$$
- $j_\eta : \mathcal{V}(M) \to \mathcal{V}(M)$ the *interior product of multivector fields by a q-form* η defined, for any $P \in \mathcal{V}^p(M)$, $p \geq q$, and $\zeta \in \Omega^{p-q}(M)$, by
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- $i_P : \Omega(M) \to \Omega(M)$ the *interior product of differential forms by a p-vector field P* defined, for any $\eta \in \Omega^q(M)$, $q \geq p$, and $Q \in \mathcal{V}^{q-p}(M)$, by

$$\langle i_P\eta, Q\rangle = (-1)^{(p-1)p/2}\langle \eta, P \wedge Q\rangle;$$

- $j_\eta : \mathcal{V}(M) \to \mathcal{V}(M)$ the *interior product of multivector fields by a q-form* η defined, for any $P \in \mathcal{V}^p(M)$, $p \geq q$, and $\zeta \in \Omega^{p-q}(M)$, by

$$\langle \zeta, j_\eta P\rangle = \langle \zeta \wedge \eta, P \wedge Q\rangle.$$

- If $p = q$,

$$j_\eta P = (-1)^{(p-1)p/2} i_P\eta = \langle \eta, P\rangle.$$

The operator Ψ

Then, for a given smooth volume form Ω on M, the interior product of p-vector fields on M, $p = 0, \ldots, m$, with Ω yields a $C^\infty(M)$-linear isomorphism

$$\begin{aligned} \Psi : \mathcal{V}^p(M) &\rightarrow \Omega^{m-p}(M) \\ P &\mapsto \Psi(P) = \Psi_P = (-1)^{(p-1)p/2} i_P \Omega. \end{aligned}$$

Its inverse map is

$$\begin{aligned} \Psi^{-1} : \Omega^{m-p}(M) &\rightarrow \mathcal{V}^p(M) \\ \eta &\mapsto \Psi^{-1}(\eta) = j_\eta \tilde{\Omega}, \end{aligned}$$

where $\tilde{\Omega}$ is the dual m-vector field of Ω, i.e., $\langle \Omega, \tilde{\Omega} \rangle = 1$.

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The operator $*$

Let

- (M, ω_0) be an almost symplectic manifold, i.e., ω_0 is a nondegenerate 2-form on M (so $m = 2n$) ;
- Λ_0 the bivector field on M defined by ω_0 ;
- $\Omega = \dfrac{\omega_0^n}{n!}$ the corresponding volume form and $\tilde{\Omega} = \dfrac{\Lambda_0^n}{n!}$ the dual $2n$-vector field of Ω.

For any $\varphi \in \Omega^p(M)$, we define the *adjoint form* $*\varphi$ *of* φ *relative to* ω_0, which is a $(2n-p)$-form, by setting

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The operator $* : \Omega^p(M) \to \Omega^{2n-p}(M)$ has the following properties :

i) $** = Id$,

ii) for any $\varphi \in \Omega^p(M)$, $\psi \in \Omega^q(M)$,

$$*(\varphi \wedge \psi) = (-1)^{(p-1)p/2}\, i_{\Lambda_0^{\#}(\varphi)}(*\psi) = (-1)^{pq+(q-1)q/2} i_{\Lambda_0^{\#}(\psi)}(*\varphi),$$

iii) $$* \frac{\omega_0^k}{k!} = \frac{\omega_0^{n-k}}{(n-k)!} \qquad (k \leq n).$$

Definition : A smooth form $\psi \in \Omega(M)$ is called *effective* if $i_{\Lambda_0}\psi = 0$ everywhere on M.

The adjoint of an effective p-form ψ ($p \leq n$) is

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Lepage's decomposition theorem

Every p-form φ on (M, ω_0) $(p \leq n)$ may be decomposed, in a unique way, as sum of simples forms :

$$\varphi = \psi_p + \psi_{p-2} \wedge \omega_0 + \ldots + \psi_{p-2q} \wedge \frac{\omega_0^q}{q!}.$$

For any $s = 0, \ldots, q$ $(q \leq [p/2])$, ψ_{p-2s} are effective and may be calculated from φ by means of iteration of the operator i_{Λ_0}.

The adjoint $*\varphi$ may be uniquely written as the sum

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Relations between Ψ and $*$ on (M, ω_0)

Since $\Lambda_0^\# : \Omega^p(M) \to \mathcal{V}^p(M)$, $p \in \mathbb{N}$, is an isomorphism, for any $P \in \mathcal{V}^p(M)$ there exists a unique $\sigma_p \in \Omega^p(M)$ s.t. $P = \Lambda_0^\#(\sigma_p)$. Therefore,

$$\Psi(P) = *\sigma_p.$$

Also, for any $\zeta \in \Omega^p(M)$ $(p \leq n)$, we have

$$\Psi^{-1}(\zeta) = \Lambda_0^\#(*\zeta). \qquad (2)$$

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Hence, by introducing the *codifferential operator* $\delta = * d *$ on (M, ω_0) and by applying the above relations for a bivector field $\Lambda = \Lambda_0^{\#}(\sigma)$ on (M, ω_0), $\sigma \in \Omega^2(M)$, we prove

Proposition

A bivector field $\Lambda = \Lambda_0^{\#}(\sigma)$ on (M, ω_0) is Poisson iff

$$2\sigma \wedge \delta(\sigma) = \delta(\sigma \wedge \sigma).$$

Remark : When $d\omega_0 = 0$,

$$2\sigma \wedge \delta(\sigma) = \delta(\sigma \wedge \sigma) \quad \Leftrightarrow \quad \{\sigma, \sigma\}_0 = 0,$$

($\{\cdot, \cdot\}_0$ being the Koszul bracket on $\Omega(M)$) which means that σ is a complementary 2-form on (M, Λ_0) in the sense of Vaisman [10].

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Poisson brackets with given Casimirs

Problem : Let M be a m-dimensional smooth manifold and $C_1, \ldots, C_{m-2k}$ smooth functions on M which are functionally independent almost everywhere. We want to construct Poisson structures Λ on M with symplectic leaves of dimension at most $2k$ which have as Casimirs the given functions $C_1, \ldots, C_{m-2k}$.

Also, we want its bracket can be written as

$$\{h_1, h_2\}\Omega = dh_1 \wedge dh_2 \wedge \Phi \Leftrightarrow \{h_1, h_2\} = \frac{dh_1 \wedge dh_2 \wedge \Phi}{\Omega},$$

where Ω is a volume element on M and Φ an $(m-2)$-form of type

$$\Phi = \left(\quad (2k-2) - \text{form} \quad \right) \wedge dC_1 \wedge \ldots \wedge dC_{m-2k}.$$

We study our problem, separately, on

- even-dimensional manifolds ;
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- even-dimensional manifolds ;
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On even-dimensional manifolds

We suppose that $\dim M = 2n$ and we consider an almost symplectic structure ω_0 on M and its associated almost Poisson tensor Λ_0 such that

$$\begin{aligned} f &= \left\langle dC_1 \wedge \ldots \wedge dC_{2n-2k}, \frac{\Lambda_0^{n-k}}{(n-k)!} \right\rangle \\ &= \left\langle \frac{\omega_0^{n-k}}{(n-k)!}, X_{C_1} \wedge \ldots \wedge X_{C_{2n-2k}} \right\rangle \neq 0 \end{aligned}$$

on an open and dense subset $\mathcal{U}$ of M, where $X_{C_i} = \Lambda_0^{\#}(dC_i)$.

That means that

$$f^2 = \det\left(\{C_i, C_j\}_0\right) \neq 0 \quad \text{on} \quad \mathcal{U}.$$

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On even-dimensional manifolds

Let

- $D = \langle X_{C_1}, \ldots, X_{C_{2n-2k}} \rangle$ the distribution on M generated by the Hamiltonians $X_{C_i} = \Lambda_0^{\#}(dC_i)$, $i = 1, \ldots, 2n-2k$,
- D° its annihilator,
- $\mathrm{orth}_{\omega_0} D$ the symplectic orthogonal of D with respect to ω_0.

Because $\det\left(\{C_i, C_j\}_0\right) = f^2 \neq 0$ on $\mathcal{U}$, at each point $x \in \mathcal{U}$, $D_x = D \cap T_x M$ is a symplectic subspace of $T_x M$ with respect to ω_{0_x}. Thus,

$$T_x M = D_x \oplus \mathrm{orth}_{\omega_{0_x}} D_x = D_x \oplus \Lambda_{0_x}^{\#}(D_x^\circ),$$

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A bivector field $\Lambda = \Lambda_0^{\#}(\sigma)$ on (M, ω_0), of rank at most $2k$, admits as unique Casimirs the functions $C_1, \ldots, C_{2n-2k}$ if and only if σ is a smooth section of $\bigwedge^2 D^\circ$ of maximal rank on $\mathcal{U}$.

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- $D = \langle X_{C_1}, \dots, X_{C_{2n-2k}} \rangle$ the distribution on M generated by the Hamiltonians $X_{C_i} = \Lambda_0^{\#}(dC_i)$, $i = 1, \dots, 2n-2k$,
- D° its annihilator,
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Now, we consider on (M, ω_0) the volume form $\Omega = \dfrac{\omega_0^n}{n!}$, a smooth section σ of $\bigwedge^2 D^\circ$ of maximal rank on $\mathcal{U}$ such that $2\sigma \wedge \delta(\sigma) = \delta(\sigma \wedge \sigma)$, and the $(2n-2)$-form

$$\Phi = \Big(-\frac{1}{f}(\sigma + \frac{g}{k-1}\omega_0) \wedge \frac{\omega_0^{k-2}}{(k-2)!} \Big) \wedge dC_1 \wedge \ldots \wedge dC_{2n-2k},$$

where $g = i_{\Lambda_0}\sigma$.

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Theorem

Under the above assumptions,

$$\Lambda = \Psi^{-1}(\Phi) \overset{(2)}{=} \Lambda_0^{\#}(*\Phi) = \Lambda_0^{\#}(\sigma)$$

is a Poisson tensor, of rank at most $2k$, for which $C_1, \dots, C_{2n-2k}$ are Casimirs. Its bracket can be calculated by

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Similar results are true on odd-dimensional manifolds.

We remark that

Any Poisson tensor Λ on M ($\dim M = 2n+1$) of rank at most $2k$, admitting $C_1, \ldots, C_{2n+1-2k}$ as Casimirs, can be viewed as a Poisson tensor on $M' = M \times \mathbb{R}$ admitting $C_1, \ldots, C_{2n+1-2k}$ and $C_{2n+2-2k} = s$ (s the canonical coordinate on $\mathbb{R}$) as Casimirs.

So, our purpose is to study our problem in the framework of M'.

We consider an almost cosymplectic structure (ϑ_0, Θ_0) on M (that means $\vartheta_0 \wedge \Theta_0^n \neq 0$ on M) and its corresponding almost Jacobi structure (Λ_0, E_0) such that

$$f = \langle dC_1 \wedge \ldots \wedge dC_{2n+1-2k},\ E_0 \wedge \frac{\Lambda_0^{n-k}}{(n-k)!} \rangle \neq 0,$$

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Let $\omega'_0 = \Theta_0 + ds \wedge \vartheta_0$ and $\Lambda'_0 = \Lambda_0 + \frac{\partial}{\partial s} \wedge E_0$ be, respectively, the almost symplectic and almost Poisson structure on $M' = M \times \mathbb{R}$ defined by (ϑ_0, Θ_0). We have

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By applying our results for manifolds of even dimension on $(M', \omega'_0, C_1, \dots, C_{2n+1-2k}, s)$, we obtain

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Under the above assumptions, let $\Omega = \vartheta_0 \wedge \frac{\Theta_0^n}{n!}$ *be the volume form on* $(M, \vartheta_0, \Theta_0)$, $(\sigma, \tau) \in \mathcal{V}^2(M) \times \mathcal{V}^1(M)$ *a pair of semi-basic forms such that*

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Remark

We remark that, in both cases (of even dimension $m = 2n$ and of odd dimension $m = 2n + 1$), when $k = 1$, the obtained brackets are of Jacobian type (1), up to a coefficient function. Precisely,

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1. Dirac brackets

Let (M, ω_0) $(\Lambda_0 = \omega_0^{-1})$ be a symplectic manifold, $\dim M = 2n$, and $C_1, \ldots, C_{2n-2k} \in C^\infty(M)$ whose the differentials are linearly independent at each point in

$$M_0 = \{x \in M \,/\, C_1(x) = 0, \ldots, C_{2n-2k}(x) = 0\}.$$

We assume that $\left(\{C_i, C_j\}_0\right)$ is invertible on an open neighborhood $\mathcal{W}$ of M_0 in M. Let c_{ij} be the coefficients of its inverse matrix which are smooth functions on $\mathcal{W}$ such that $\sum_{j=1}^{2n-2k} \{C_i, C_j\}_0 c_{jk} = \delta_{ik}$. We consider on $\mathcal{W}$ the 2-form

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We can easily prove that σ is a section of $\bigwedge^2 D^\circ$ (D° being the annihilator of $D = \langle X_{C_1}, \ldots, X_{C_{2n-2k}} \rangle$) of maximal rank on $\mathcal{W}$ which verifies $2\sigma \wedge \delta(\sigma) = \delta(\sigma \wedge \sigma)$. Thus,

$$\Lambda = \Lambda_0^{\#}(\sigma) = \Lambda_0 + \sum_{i<j} c_{ij} X_{f_i} \wedge X_{f_j}$$

defines a Poisson structure on $\mathcal{W}$ whose corresponding bracket $\{\cdot,\cdot\}$ on $C^\infty(\mathcal{W})$ is given, for any $h_1, h_2 \in C^\infty(\mathcal{W})$, by

$$\{h_1, h_2\}\Omega = \frac{1}{f} dh_1 \wedge dh_2 \wedge \frac{\omega_0^{k-1}}{(k-1)!} \wedge dC_1 \wedge \ldots \wedge dC_{2n-2k}. \quad (3)$$

In the above expression of Λ we recognize the Poisson structure defined by Dirac on an open neighborhood $\mathcal{W}$ of the constrained submanifold M_0 of M and in (3) a new expression of the Dirac bracket.

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defines a Poisson structure on $\mathcal{W}$ whose corresponding bracket $\{\cdot, \cdot\}$ on $C^\infty(\mathcal{W})$ is given, for any $h_1, h_2 \in C^\infty(\mathcal{W})$, by

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In the above expression of Λ we recognize the Poisson structure defined by Dirac on an open neighborhood $\mathcal{W}$ of the constrained submanifold M_0 of M and in (3) a new expression of the Dirac bracket.

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2. Periodic Toda and Volterra lattices

We consider the linear Poisson structure Λ_T associated with the periodic Toda lattice of $n = 3$ particles. This Poisson structure has two well-known Casimir functions. Using our results we construct another Poisson structure having the same Casimir invariants with Λ_T.

The periodic Toda lattice of $n = 3$ particles is the system of ordinary differential equations on $\mathbb{R}^6$ which in Flaschka's coordinate system $(a_1, a_2, a_3, b_1, b_2, b_3)$ takes the form

$$\dot{a}_i = a_i(b_{i+1} - b_i), \quad \dot{b}_i = 2(a_i^2 - a_{i-1}^2) \quad (i \in \mathbb{Z} \quad (a_{i+3}, b_{i+3}) = (a_i, b_i)).$$

It is hamiltonian with respect to the Lie-Poisson structure

$$\Lambda_T = a_1 \frac{\partial}{\partial a_1} \wedge (\frac{\partial}{\partial b_1} - \frac{\partial}{\partial b_2}) + a_2 \frac{\partial}{\partial a_2} \wedge (\frac{\partial}{\partial b_2} - \frac{\partial}{\partial b_3}) + a_3 \frac{\partial}{\partial a_3} \wedge (\frac{\partial}{\partial b_3} - \frac{\partial}{\partial b_1}),$$

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Examples

which is of rank 4 on $\mathcal{U} = \{(a, b) \in \mathbb{R}^6 \,/\, a_1 a_2 + a_1 a_3 + a_2 a_3 \neq 0\}$ and it admits two Casimirs :

$$C_1 = b_1 + b_2 + b_3 \qquad \text{and} \qquad C_2 = a_1 a_2 a_3.$$

We consider on $\mathbb{R}^6$ the symplectic form $\omega_0 = \sum_{i=1}^{3} da_i \wedge db_i$, its associated Poisson tensor $\Lambda_0 = \sum_{i=1}^{3} \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial b_i}$, and the corresponding volume element

$$\Omega = \frac{\omega_0^3}{3!} = da_1 \wedge db_1 \wedge da_2 \wedge db_2 \wedge da_3 \wedge db_3.$$

We have

$$f = \langle dC_1 \wedge dC_2, \Lambda_0 \rangle = -(a_1 a_2 + a_2 a_3 + a_1 a_3) \neq 0 \quad \text{on} \quad \mathcal{U}.$$

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The hamiltonian vector fields of C_1 and C_2 with respect to Λ_0 :

$$X_{C_1} = -\left(\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}\right), \quad X_{C_2} = a_2 a_3 \frac{\partial}{\partial b_1} + a_1 a_3 \frac{\partial}{\partial b_2} + a_1 a_2 \frac{\partial}{\partial b_3}.$$

So, $D = \langle X_{C_1}, X_{C_2} \rangle$ and

$$D^\circ = \Big\{ \sum_{i=1}^{3} (\alpha_i da_i + \beta_i db_i) \in \Omega^1(\mathbb{R}^6) \ /$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \quad \text{and} \quad a_1 a_2 \beta_3 + a_1 \beta_2 a_3 + \beta_1 a_2 a_3 = 0 \Big\}.$$

The family of 1-forms $(\sigma_1, \sigma_2, \sigma_1', \sigma_2')$,

$$\sigma_1 = da_1 - da_2, \quad \sigma_2 = da_2 - da_3,$$
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Examples

The section σ_T of $\bigwedge^2 D^\circ \to \mathcal{U}$, which corresponds to Λ_T via $\Lambda_0^\#$, the function $g_T = i_{\Lambda_0}\sigma_T$ and the 4-form Φ_T are written as

$$\sigma_T = \sigma_1 \wedge (\sigma_1' + \sigma_2') + \sigma_2 \wedge \sigma_2', \quad g_T = i_{\Lambda_0}\sigma_T = -(a_1 + a_2 + a_3),$$

$$\begin{aligned} \Phi_T &= -f^{-1}(\sigma_T + g_T\omega_0) \wedge dC_1 \wedge dC_2 \\ &= -a_1 db_1 \wedge da_2 \wedge da_3 \wedge db_3 + a_1 da_2 \wedge db_2 \wedge da_3 \wedge db_3 \\ &\quad +a_2 da_1 \wedge db_1 \wedge da_3 \wedge db_3 - a_2 da_1 \wedge db_1 \wedge db_2 \wedge da_3 \\ &\quad +a_3 da_1 \wedge da_2 \wedge db_2 \wedge db_3 + a_3 da_1 \wedge db_1 \wedge da_2 \wedge db_2. \end{aligned}$$

Thus,

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and all other brackets are zero.

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Examples

Now, we consider on $\mathbb{R}^6$ the 2-form

$$\sigma = \sigma_1 \wedge \sigma_2 + \sigma_1' \wedge \sigma_2'.$$

We have $2\sigma \wedge \delta(\sigma) = \delta(\sigma \wedge \sigma)$. So, $\Lambda_0^{\#}(\sigma)$ is Poisson of rank 4 on $\mathcal{U}$. Also, $g = i_{\Lambda_0}\sigma = 0$ and

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Thus,

$$\begin{aligned}\{a_1, a_2\}\Omega &= da_1 \wedge da_2 \wedge \Phi = a_1 a_2 \Omega, \\ \{a_1, a_3\}\Omega &= da_1 \wedge da_3 \wedge \Phi = -a_1 a_3 \Omega, \\ \{a_2, a_3\}\Omega &= da_2 \wedge da_3 \wedge \Phi = a_2 a_3 \Omega, \\ \{b_1, b_2\}\Omega &= db_1 \wedge db_2 \wedge \Phi = \Omega, \\ \{b_1, b_3\}\Omega &= db_1 \wedge db_3 \wedge \Phi = -\Omega, \\ \{b_2, b_3\}\Omega &= db_2 \wedge db_3 \wedge \Phi = \Omega,\end{aligned}$$

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Examples

It turns out that this structure decomposes as a direct sum of two Poisson structures :

$$\Lambda = a_1 a_2 \frac{\partial}{\partial a_1} \wedge \frac{\partial}{\partial a_2} - a_1 a_3 \frac{\partial}{\partial a_1} \wedge \frac{\partial}{\partial a_3} + a_2 a_3 \frac{\partial}{\partial a_2} \wedge \frac{\partial}{\partial a_3} + \frac{\partial}{\partial b_1} \wedge \frac{\partial}{\partial b_2} - \frac{\partial}{\partial b_1} \wedge \frac{\partial}{\partial b_3} + \frac{\partial}{\partial b_2} \wedge \frac{\partial}{\partial b_3},$$

the first of which (involving only the *a* variables in Flaschka's coordinates) is the quadratic Poisson bracket associated to the Volterra lattice (also known as the KM-system) :

$$\dot{a}_i = a_i(a_{i+1} - a_{i-1}) \qquad (i \in \mathbb{Z}, \quad a_{i+3} = a_i)$$

with Hamiltonian $H = a_1 + a_2 + a_3$.

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Thank you for your attention !