CIRCLE VALUED MOMENTUM MAPS AND FIXED POINTS OF SYMPLECTIC ACTIONS

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PLAN OF THE PRESENTATION

- ***Every symplectic circle action admits either a momentum map*** $\mathrm{J} : M \to \mathbb{R}$ ***or a circle valued momentum map*** $\mu : M \to S^1$ ***relative to some (possibly different) invariant symplectic form.***

- $\mathrm{J} : M \to \mathbb{R}$ ***is Morse-Bott and map*** $\mu : M \to S^1$ ***is Morse-Bott-Novikov. Each connected component of the fixed point set has even index.***

- ***Equivariant Darboux; Melbourne-Dellnitz***

- ***Detect Hamiltonian flows by fixed points***

PRELIMINARIES

THE CIRCLE $\mathbb{R}/\mathbb{Z}$

Circle $S^1 \equiv \mathbb{R}/\mathbb{Z}$, $\pi : \mathbb{R} \ni t \mapsto [t] \in \mathbb{R}/\mathbb{Z}$ the canonical projection, a surjective submersive Lie group homomorphism.
$T_0\pi : r \in \mathbb{R} \overset{\sim}{\longmapsto} T_0\pi(r) \in T_{[0]}(\mathbb{R}/\mathbb{Z})$ isomorphism
L_t, $L_{[t]}$ left (equiv. right) translation on $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$; $\pi \circ L_t = L_{[t]} \circ \pi$
$T_{[t]}(\mathbb{R}/\mathbb{Z}) = \{T_t\pi(r) \mid r \in \mathbb{R}\}$, $\left(T_{[0]}L_{[t]} \circ T_0\pi\right)(0, r) = T_t\pi(t, r)$, $t, r \in \mathbb{R}$
π is also the exponential map

Length form $\lambda \in \Omega^1(\mathbb{R}/\mathbb{Z})$ defined by $\lambda([t])\,(T_t\pi(r)) := r$. Since in local coordinates $T_t\pi(r) = r\frac{\partial}{\partial t} \Rightarrow \lambda = \mathbf{d}t$. Therefore, $\int_{\mathbb{R}/\mathbb{Z}} \lambda = \int_0^1 \mathbf{d}t = 1$ and λ is left (equivalently, right) invariant.

LOGARITHMIC EXTERIOR DIFFERENTIAL

$f : M \xrightarrow{C^\infty} \mathbb{R}/\mathbb{Z}$, **logarithmic exterior differential** $\boldsymbol{\delta} f \in \Omega^1(M)$

$$\boldsymbol{\delta} f(m)(v_m) := T_{f(m)}L_{-f(m)}\left(T_m f(v_m)\right) \in \mathbb{R}$$

If $X \in \mathfrak{X}(M)$, define $\langle \boldsymbol{\delta} f, X\rangle \in C^\infty(M)$ by $\langle \boldsymbol{\delta} f, X\rangle (m) := \boldsymbol{\delta} f(m)(X(m))$ for any $m \in M$. $\boldsymbol{\delta}(fg) = \boldsymbol{\delta} f + \boldsymbol{\delta} g$.

Useful formula: $f^*\lambda = \boldsymbol{\delta} f$.

Relation of logarithmic exterior differential to exterior differential

$\widetilde{M} := \{(m,t) \in M \times \mathbb{R} \mid f(m) = [t]\}$ pull back bundle by f of the principal $\mathbb{Z}$-bundle $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. So $\tilde{\pi} : \widetilde{M} \ni (m,t) \mapsto m \in M$ is also a principal $\mathbb{Z}$-bundle and hence a covering space.

$$T_{(m,t)}\widetilde{M} = \{(v_m, (t, \boldsymbol{\delta} f(m)(v_m)) \mid f(m) = [t]\}$$

Canonical lift $\tilde{f} : \widetilde{M} \ni (m,t) \mapsto t \in \mathbb{R}$ of f; $\pi \circ \tilde{f} = f \circ \tilde{\pi}$, so

$$(*)\ \boldsymbol{\delta} f(m)(v_m) = \mathbf{d}\tilde{f}(m,t)\,(v_m, (t, \boldsymbol{\delta} f(m)(v_m)))\,,\ f(m) = [t],\ v_m \in T_mM$$

$m \in M$ is a **critical point of** f ($T_mf = 0 \Leftrightarrow \boldsymbol{\delta} f(m) = 0$) if and only if all $(m,t) \in \tilde{\pi}^{-1}(m) \subset \widetilde{M}$ are critical points of the real valued function $\tilde{f}$.

$\mathrm{Crit}(f) := \{m \in M \mid \boldsymbol{\delta} f(m) = 0\}$ the set of critical points of f.

HESSIAN

1.) Recall that if $f : M \to \mathbb{R}$ is smooth and $\mathbf{d}f(m_0) = 0$, the **Hessian** $(\mathrm{Hess}\, f)(m_0)$: $(\mathrm{Hess}\, f)(m_0) : T_{m_0}M \times T_{m_0}M \to \mathbb{R}$ is defined by

$$(\mathrm{Hess}\, f)(m_0)(u,v) := \boldsymbol{\pounds}_{\widetilde{u}} \left(\langle \mathbf{d}f, \widetilde{v}\rangle\right)(m_0) = \langle \mathbf{d}\left(\langle \mathbf{d}f, \widetilde{v}\rangle\right)(m_0), u\rangle\,,$$

$\forall u, v \in T_{m_0}M$, where $\widetilde{u}, \widetilde{v}$ are arbitrary local smooth vector fields in a neighborhood of m_0 such that $\widetilde{u}(m_0) = u$, $\widetilde{v}(m_0) = v$. The Hessian depends only on u, v (not on $\widetilde{u}$, $\widetilde{v}$); symmetric bilinear form.

2.) $f : M \to \mathbb{R}/\mathbb{Z}$ smooth and $\boldsymbol{\delta} f(m_0) = 0$. The **Hessian** $(\mathrm{Hess}\, f)(m_0)$: $(\mathrm{Hess}\, f)(m_0) : T_{m_0}M \times T_{m_0}M \to \mathbb{R}$ is defined by

$$(\mathrm{Hess}\, f)(m_0)(u,v) := \boldsymbol{\pounds}_{\widetilde{u}} \left(\langle \boldsymbol{\delta} f, \widetilde{v}\rangle\right)(m_0) = \langle \mathbf{d}\left(\langle \boldsymbol{\delta} f, \widetilde{v}\rangle\right)(m_0), u\rangle\,,$$

$\forall u, v \in T_{m_0}M$, $\widetilde{u}, \widetilde{v}$ any local smooth vector fields in a neighborhood of m_0 such that $\widetilde{u}(m_0) = u$, $\widetilde{v}(m_0) = v$. If $m_0 \in \mathrm{Crit}(f)$, then

$$(\mathrm{Hess}\, f)(m_0)(u,v) = (\mathrm{Hess}\, \tilde{f})(m_0, t_0)\Big((u, (t_0, 0)), (v, (t_0, 0))\Big),$$

for any $t_0 \in \mathbb{R}$ satisfying $f(m_0) = [t_0]$, $u, v \in T_{m_0}M$.

Proof: Recall that at any $(m,t) \in \widetilde{M}$, the tangent space is

$$T_{(m,t)}\widetilde{M} = \{(v_m, (t, \boldsymbol{\delta} f(m)(v_m)) \mid f(m) = [t]\}$$

Thus, if $\boldsymbol{\delta} f(m_0) = 0$ and $(m_0, t_0) \in \widetilde{\pi}^{-1}(m_0)$ then

$$T_{(m_0,t_0)}\widetilde{M} = \{(v_{m_0}, (t_0, 0) \mid f(m_0) = [t_0]\}$$

Let $v \in T_{m_0}M$, $\widetilde{v}$ arbitrary local smooth vector field defined in a neighborhood of m_0 and satisfying $\widetilde{v}(m_0) = v$. Then $(\widetilde{v}, \langle \boldsymbol{\delta} f, \widetilde{v}\rangle)$ is a smooth local vector field defined in a neighborhood of $(m_0, t_0) \in \widetilde{M}$ whose value at (m_0, t_0) is $(v, (t_0, 0)) \in T_{(m_0,t_0)}\widetilde{M}$. So, if $m(\varepsilon) \in M$ with $m(0) = m_0$ and $m'(0) = u$,

$$\begin{aligned}
&(\operatorname{Hess} \widetilde{f})(m_0, t_0)\Big((u, (t_0, 0)), (v, (t_0, 0))\Big) \\
&\qquad = \left\langle \mathbf{d} \left\langle \mathbf{d}\widetilde{f}, (\widetilde{v}, \langle \boldsymbol{\delta} f, \widetilde{v}\rangle)\right\rangle (m_0, t_0), (u, (t_0, 0))\right\rangle \\
&\qquad = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left\langle \mathbf{d}\widetilde{f}, (\widetilde{v}, \langle \boldsymbol{\delta} f, \widetilde{v}\rangle)\right\rangle (m(\varepsilon), t_0) \overset{(*)}{=} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \langle \boldsymbol{\delta} f, \widetilde{v}\rangle (m(\varepsilon)) \\
&\qquad = \langle \mathbf{d} (\langle \boldsymbol{\delta} f, \widetilde{v}\rangle) (m_0), u\rangle = (\operatorname{Hess} f)(m_0)(u, v).
\end{aligned}$$

So $(\operatorname{Hess} f)(m_0)$ is well defined and is a symmetric bilinear form.□

$m_0 \in M$ is **non-degenerate** if $(\mathrm{Hess}\, f)(m_0)$ is a non-degenerate bilinear form. So, *m_0 is a non-degenerate critical point of f if and only if all $(m_0, t_0) \in \tilde{\pi}^{-1}(m_0) \subset \widetilde{M}$ are non-degenerate critical points of $\tilde{f}$*. The Morse Lemma for $\tilde{f}$ and the fact that $\tilde{\pi} : \widetilde{M} \to M$ is a covering space, implies that non-degenerate critical points of $f : M \to \mathbb{R}/\mathbb{Z}$ are isolated. In particular, if M is compact, then there are only finitely many non-degenerate critical points of f.

MORSE-BOTT-NOVIKOV MAPS

1.) $f : M \to \mathbb{R}$ is **Morse** if all its critical points are non-degenerate.

2.) $f : M \to \mathbb{R}$ is **Morse-Bott** if the critical set $\mathrm{Crit}(f)$ of f is a disjoint union of connected submanifolds C_i of M such that $\ker(\mathrm{Hess}\, f)(m) = T_m C_i$, for each i and $m \in C_i$.

The **index** of m is the number of negative eigenvalues of $(\mathrm{Hess}\, f)(m)$.

3.) $f : M \to \mathbb{R}/\mathbb{Z}$ is **Morse-Bott-Novikov** if the critical set Crit$(f) := \{m \in M \mid \boldsymbol{\delta} f(m) = 0\}$ of f is a disjoint union of connected submanifolds C_i of M such that ker(Hess $f)(m) = T_m C_i$, for each i and $m \in C_i$.

The **index** of m is the number of negative eigenvalues of (Hess $f)(m)$. Since Crit(f) is closed, if M is compact, then it has only a finite number of connected components.

$(\mathrm{Hess}\,\tilde{f})(m_0, t_0)\Big((u, (t_0, 0)), (v, (t_0, 0))\Big) = (\mathrm{Hess}\, f)(m_0)(u, v)$ implies that Crit $\tilde{f} = \tilde{\pi}^{-1}$(Crit f).

Thus, from $\boldsymbol{\delta} f(m)(v_m) = \mathbf{d}\tilde{f}(m, t)\Big(v_m, (t, \boldsymbol{\delta} f(m)(v_m))\Big)$, $f(m) = [t]$, and the above formula for the Hessian, we conclude that:

$f : M \to \mathbb{R}/\mathbb{Z}$ is Morse-Bott-Novikov if and only if $\tilde{f} : \widetilde{M} \to \mathbb{R}$ is Morse-Bott.

THE CIRCLE VALUED MOMENTUM MAP

(M, σ) symplectic: $\sigma \in \Omega^2(M)$ non-degenerate, $\mathbf{d}\sigma = 0$.

$\Phi : \mathbb{R}/\mathbb{Z} \times M \to M$ be a smooth action, $\Phi_{[t]}^* \sigma = \sigma$, $\forall [t] \in \mathbb{R}/\mathbb{Z}$.

For $r \in \mathbb{R}$, the **infinitesimal generator** is defined by

$$r_M(x) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Phi_{[r\varepsilon]}(x).$$

Action Φ is **Hamiltonian** if there exists a smooth map $\mu \colon M \to \mathbb{R}$, called the **momentum map**, such that $\mathbf{i}_{1_M}\sigma = \sigma(1_M, \cdot) = \mathbf{d}\mu$. Existence of $\mu \Leftrightarrow$ to the exactness of $\mathbf{i}_{1_M}\sigma$. So, obstruction to Φ being Hamiltonian lies in $H^1(M; \mathbb{R})$; thus, if $H^1(M; \mathbb{R}) = 0$ is the trivial group then every symplectic $\mathbb{R}/\mathbb{Z}$-action on M is Hamiltonian.

A **circle valued momentum map** $\mu : M \to \mathbb{R}/\mathbb{Z}$ is defined by the condition $\mu^*\lambda = \mathbf{i}_{1_M}\sigma$, where $\lambda \in \Omega^1(\mathbb{R}/\mathbb{Z})$ the standard length form.

Remarks

1.) $\Phi : G \times M \to M$, $\Phi_g^* \sigma = \sigma$, is **Hamiltonian** if $\exists \mathbf{J} : M \to \mathfrak{g}^*$ such that $X_{\mathbf{J}^\xi} = \xi_M$, where $\xi_M(m) := \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(t\xi)}(m)$ and $\mathbf{J}$ is equivariant relative to the coadjoint action. If G is compact, equivariance can always be achieved by averaging from some momentum map. Obstruction to existence: iff the following map vanishes

$$\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = H^1(\mathfrak{g},\mathbb{R}) \ni [\xi] \longmapsto [\xi_M] \in \mathfrak{X}_\omega(M)/\operatorname{Ham}(M) \cong H^1_{deR}(M,\mathbb{R})$$

2.) The definition is equivalent to that of group valued momentum maps in the case of $\mathbb{R}/\mathbb{Z}$ because of formula $\mu^*\lambda = \boldsymbol{\delta}\mu$.

We recall the definition only in the case of Abelian Lie groups because only this situation is interesting for us. G Abelian Lie group, $\mathfrak{g}$ its Lie algebra, $(\cdot,\cdot)$ bilinear symmetric nondegenerate form on $\mathfrak{g}$, (M,ω) symplectic, G-action is canonical. Then $\mathbf{J} : M \to G$ is a G-***valued momentum map*** if for all $\xi \in \mathfrak{g}$, $m \in M$, $v_m \in T_mM$

$$\left(\mathbf{i}_{\xi_M}\omega\right)(m)(v_m) = \left(T_m\left(L_{\mathbf{J}(m)^{-1}} \circ \mathbf{J}\right)(v_m), \xi\right).$$

For non-Abelian Lie groups, group valued momentum maps are defined on spaces that are neither symplectic nor Poisson; theory of **quasi-Hamiltonian spaces**.

3.) Notion of **cylinder valued momentum map** CoDaMo[1988]. (M, ω) connected paracompact symplectic acted upon canonically by $\mathfrak{g}$. $\pi : M \times \mathfrak{g}^* \to M$ is a principal $(\mathfrak{g}^*, +)$-bundle: $\nu \cdot (m, \mu) := (m, \mu - \nu)$. $\alpha \in \Omega^1(M \times \mathfrak{g}^*; \mathfrak{g}^*)$ connection one-form defined by

$$\langle \alpha(m, \mu)\,(v_m, \nu)\,, \xi \rangle := \left(\mathbf{i}_{\xi_M}\omega\right)(m)(v_m) - \langle \nu \xi \rangle$$

Vertical bundle: $V(m, \mu) := \left\{(0, \rho) \in T_{(m,\mu)}(M \times \mathfrak{g}^*) \mid \rho \in \mathfrak{g}^*\right\}$
Horizontal bundle:
$H(m, \mu) := \left\{(v_m, \nu) \in T_{(m,\mu)}(M \times \mathfrak{g}^*) \mid \left(\mathbf{i}_{\xi_M}\omega\right)(m)(v_m) = \langle \nu, \xi \rangle\,, \forall \xi \in \mathfrak{g}\right\}$

α is a flat connection.

For $(x, \mu) \in M \times \mathfrak{g}^*$, let $(M \times \mathfrak{g}^*)(z, \mu)$ be the **holonomy bundle** $=$ the set of points in $M \times \mathfrak{g}^*$ which can be joined to z by a horizontal curve.

$\mathcal{H}(z,\mu)$ **holonomy group** based at (z,μ): horizontally lift a loop in M to a curve $c(t)$ with $c(0) = (z,\mu)$; then $c(1) = \nu \cdot (z,\mu) = (z, \mu - \nu)$. By Ambrose-Singer: $\mathcal{H}(z,\mu)$ is discrete iff α is flat. This is equivalent to: horizontal subbundle is an involutive distribution having the holonomy bundles as maximal integral manifolds. Then the **holonomy reduced bundle** $\pi : (M \times \mathfrak{g}^*)(z,\mu) \to M$, a principal bundle with group $\mathcal{H}(z,\mu)$, is a covering map.

Denote $\widetilde{M} := (M \times \mathfrak{g}^*)(z,\mu)$, $\widetilde{p} := \pi|_{\widetilde{M}} : \widetilde{M} \to M$, $\mathcal{H} = \mathcal{H}(z,\mu)$.

Let $\widetilde{\mathbf{K}} : \widetilde{M} \subset M \times \mathfrak{g}^* \to \mathfrak{g}^*$ projection.

$\overline{\mathcal{H}}$ closure of $\mathcal{H}$ in $\mathfrak{g}^* \Longrightarrow C := \mathfrak{g}^*/\overline{\mathcal{H}} \cong \mathbb{R}^a \times \mathbb{T}^b$ is a cylinder.

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\mathbf{K}}} & \mathfrak{g}^* \\ {\scriptstyle \widetilde{p}}\downarrow & & \downarrow{\scriptstyle \pi_C} \\ \mathfrak{g}^* & \xrightarrow{\mathbf{K}} & \mathfrak{g}^*/\overline{\mathcal{H}} \end{array}$$

So $\mathbf{K}(m) := \pi_C(\nu)$, where $\nu \in \mathfrak{g}^*$ is any element such that $(m,\nu) \in \widetilde{M}$. $\mathbf{K}$ is the **cylinder valued momentum map**. It has the Noether property. $\mathcal{H}$ is not closed, in general. Action admits a standard momentum map iff $\mathcal{H} = 0$ and $\mathbf{K}$ is it.

Two holonomy bundles $\widetilde{M}_1$, $\widetilde{M}_2 = \left\{(m, \nu + \tau) \mid (m, \nu) \in \widetilde{M}_1\right\}$ for some $\tau \in \mathfrak{g}^*$. So, $\mathbf{K}_{\widetilde{M}_2} = \mathbf{K}_{\widetilde{M}_1} + \pi_C(\tau)$.

Example: $(\mathbb{T}^2, \mathbf{d}\theta_1 \wedge \mathbf{d}\theta_2)$ and $S^1 = \{e^{i\phi}\}$ acts canonically by $e^{i\phi} \cdot \left(e^{i\theta_1}, e^{i\theta_2}\right) := \left(e^{i(\theta_1+\phi)}, e^{i\theta_2}\right)$. Holonomy group at any point is $(\mathbb{Z}, +)$ and holonomy bundles are

$$\widetilde{\mathbb{T}^2}_\tau := \left\{\left(\left(e^{i\theta_1}, e^{i\theta_2}\right), \tau + \theta_2\right) \in \mathbb{T}^2 \times \mathbb{R} \mid \theta_1, \theta_2\right\}$$

So $\mathbf{K} : \mathbb{T}^2 \ni \left(e^{i\theta_1}, e^{i\theta_2}\right) \mapsto e^{i\theta_2} \in S^1 \cong \mathbb{R}/\mathbb{Z}$ is the cylinder valued momentum map.

4.) Any cylinder valued momentum map for an Abelian Lie algebra action whose associated holonomy group is closed can be understood as a Lie group valued momentum map. Indeed, $f : \mathfrak{g} \ni \xi \mapsto (\xi, \cdot) \in \mathfrak{g}^*$ is an isomorphism. Let $\mathcal{T} := f^{-1}(\mathcal{H}) \subset \mathfrak{g}$. Then $\bar{f} : \mathfrak{g}/\mathcal{T} \ni \xi + \mathcal{T} \mapsto (\xi, \cdot) + \mathcal{H} \in \mathfrak{g}^*/\mathcal{H}$ is also an isomorphism of Abelian Lie groups. Suppose $\mathcal{H}$ is closed. Then $\mathbf{J} := \bar{f}^{-1} \circ \mathbf{K} : M \to \mathfrak{g}/\mathcal{T}$ is a $\mathfrak{g}/\mathcal{T}$-valued momentum map.

Conversely, give hypotheses ensuring that a Lie group valued momentum map naturally induces a cylinder valued momentum map.

Let G be a connected Abelian Lie group, $\exp : \mathfrak{g} \to G$. Suppose that $\exists \mathbf{A} : M \to G$, group valued momentum map (using $(\cdot, \cdot)$). Then $\mathcal{H} \subseteq f(\ker \exp)$, $\mathcal{H} = \overline{\mathcal{H}}$, $\mathcal{H}$ is discrete (because $\ker \exp$ is). Let $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map and $\mathbf{J} : \bar{f}^{-1} \circ \mathbf{K} : M \to \mathfrak{g}/\mathcal{T}$. If $f(\ker \exp) \subseteq \mathcal{H} \Leftrightarrow f(\ker \exp) = \mathcal{H}$, then $\mathcal{T} = \ker \exp$ and $\mathbf{J} : M \to \mathfrak{g}/\mathcal{T} \cong G$ is a G-valued momentum map that differs from $\mathbf{A}$ by a constant in G.

Conversely, if $\mathcal{H} = f(\ker \exp)$, then $\mathbf{J} : M \to \mathfrak{g}/\ker \exp \cong G$ is a G-valued momentum map.

All proofs in OrRa[2004].

THE MAIN THEOREM

Let the circle $\mathbb{R}/\mathbb{Z}$ act symplectically on the compact symplectic manifold (M, σ). Denote by $M^{\mathbb{R}/\mathbb{Z}}$ the fixed point set of the $\mathbb{R}/\mathbb{Z}$-action. Then either the action admits a standard momentum map or, if not, there exists a $\mathbb{R}/\mathbb{Z}$-invariant symplectic form ω on M that admits a circle valued momentum map $\mu : M \to \mathbb{R}/\mathbb{Z}$. Moreover, μ is a Morse-Bott-Novikov function and each connected component of $M^{\mathbb{R}/\mathbb{Z}} = \mathrm{Crit}(\mu)$ has even index.

1.) If $[\sigma] \in H^1(M, \mathbb{Z}) \Longrightarrow \omega = \sigma$.

2.) ω is close to σ and $\exists k \in \mathbb{R}$ such that $k[\mathbf{i}_{1_M}\omega] \in H^1(M; \mathbb{Z})$ for some $k \in \mathbb{R}$. There is always a corresponding circle valued momentum map $\mu \colon M \to S^1$ for any such form $k\omega$; due to McDuff[1988].

3.) Frankel's theorem [1959] implies that the momentum map for a circle action on a compact Kähler manifold is Morse-Bott and the index of each connected component of the fixed point set of the action is even. Second part extends this theorem.

3.) The first modern definition of the momentum map is due to Kostant (1965 Phillips lectures at Haverford, written by Dale Husemoller, and in the 1965 U.S. Japan Seminar [1966]), and a few months later to Souriau (1965 Marseille lecture notes, [1966]) who recognized its physical significance. “Momentum" versus "moment"; mistranslation pointed out by Duistermaat. Translation of Souriau’s book uses "momentum". Look at
http://en.wikipedia.org/wiki/Torque

In physics: torque = moment = moment of momentum

$$\frac{d}{dt} momentum = moment$$

In engineering: "Moment" is the general term for the tendency of one or more applied forces to rotate an object about an axis (= “torque = moment" in physics). "Torque" is a special case of this.

A “couple" is a system of forces with a resultant moment but no resultant force, sometimes “pure moment": creates rotation without translation, or more generally without any acceleration of the center of mass. The resultant moment of a couple is called a “torque".

4.) However, Frankel [1959] has the first modern definition of the momentum map for S^1.

5.) If $\mu : M \to \mathbb{R}$ is a standard momentum map for a circle action on a $2n$-dimensional compact symplectic manifold (M, σ), it is well-known that it has at least $n+1$ critical points or, equivalently, the circle action has at least $n+1$ fixed points.

Proof: μ is Morse-Bott implies that the connected components C_i of Crit(μ) are submanifolds of M.

If $\exists \dim C_i > 0 \Longrightarrow$ infinitely many critical points of μ; result obvious.

If $\forall \dim C_i = 0 \Longrightarrow \mu$ is a Morse function $\Longrightarrow \mu$ is perfect (i.e., the Morse inequalities are equalities) because of classical result: If f is a Morse function on a compact manifold whose critical points have only even indices, then it is a perfect Morse function.

$m_k(\mu) :=$ #critical points of μ of index k

So, total number of critical points of μ equals

$$\sum_{k=0}^{2n} m_k(\mu) = \sum_{k=0}^{2n} \mathsf{b}_k(M),$$

where $\mathsf{b}_k(M) := \dim\left(H^k(M, \mathbb{R})\right)$ is the kth Betti number of M.

σ symplectic form $\Rightarrow [\sigma^k] \in H^{2k}(M, \mathbb{R})$ nontrivial for $k = 0, \ldots, n$, and hence $\mathsf{b}_{2k}(M) \geq 1$, which then implies that the total number of critical points of μ is at least $n + 1$. $\square$

6.) Can one get the same result for circle valued momentum maps? NO! This is one of the outstanding problems in the topology of the momentum map. Let's try to see why. Idea: Replace Morse inequalities by the Novikov inequalities if all critical points are non-degenerate. Possible because there is a Morse-Bott-Novikov theory.

Number of critical points of the circle-valued momentum map μ is $\sum_{k=0}^{2n} m_k(\mu)$ which is estimated from below by

$$\sum_{k=0}^{2n} \left(\widehat{b}_k(M) + \widehat{q}_k(M) + \widehat{q}_{k-1}(M)\right),$$

where $\widehat{b}_k(M)$ is the rank of the $\mathbb{Z}((t))$-module $H_k(\widetilde{M}, \mathbb{Z}) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}((t))$, $\widehat{q}_k(M)$ is the torsion number of this module, and $\widetilde{M}$ is the pull back by $\mu : M \to \mathbb{R}/\mathbb{Z}$ of the principal $\mathbb{Z}$-bundle $t \in \mathbb{R} \mapsto [t] \in \mathbb{R}/\mathbb{Z}$.

Unfortunately, this lower bound can be zero! For example, the circle action on the two-torus by rotation on the first factor is free and hence has no fixed points. Farber [2004] §7.3 has more information.

PeTo [2010]: If lower bound is strictly positive, then it must be at least two. If dim$(M) \geq 8$, then if the lower bound is strictly positive, it must be at least three. Proved using localization in equivariant cohomology. Lower bound is at least $n+1$ provided that the so-called Chern class map is somewhere injective. No examples our counterexamples. No universal lower bound for non-Hamiltonian symplectic circle actions with at least one fixed point is available.

7.) Many examples of interesting Hamiltonian circle actions; e.g., Karshon's [1999] classification in dimension 4.

Many situations in geometry and dynamical systems when one has a symplectic circle action (equivalently, a symplectic periodic flow) on a manifold but the one-form $\mathbf{i}_{\xi_M}\sigma$ is not exact, e.g., consider any action without fixed points such as a free action.

Duistermaat-Pelayo [2007], Pelayo [2010] give infinitely many examples of compact connected symplectic manifolds in any dimension equipped with symplectic *free* torus actions that are hence not Hamiltonian. Famous example is the *Kodaira variety* [1964], also known as the *Kodaira-Thurston manifold* (Example 3.8 on page 88 of McDuff-Salamon [1998]), which was pointed out by Thurston [1978] dtto be a non-Kähler symplectic manifold.

TECHNICAL LEMMA

$\Phi : (\mathbb{R}/\mathbb{Z}) \times M \to M$ smooth action, $\varphi \colon \mathbb{R}/\mathbb{Z} \to N$ smooth map. Define $\psi : (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \to N$ by

$$\psi([s],[t]) := \Phi_{[t]}\left(\varphi\left([s]\right)\right).$$

Then, if $\alpha \in \Omega^2(M)^{\mathbb{R}/\mathbb{Z}}$ (i.e., $(\mathbb{R}/\mathbb{Z})$-invariant 2-forms), we have

$$\int_{(\mathbb{R}/\mathbb{Z})\times(\mathbb{R}/\mathbb{Z})} \psi^*\alpha = -\int_{\mathbb{R}/\mathbb{Z}} \varphi^*(\mathbf{i}_{1_M}\alpha).$$

Proof: Let $\beta \in \Omega^2((\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}))$. Denote by $[s]$ the elements of the first circle and by $[t]$ those of the second. Let $\partial/\partial t \in \mathfrak{X}(\mathbb{R}/\mathbb{Z})$ be the left (equivalently, right) invariant vector field whose value at [0] is 1. Let $\mathbf{d}t \in \Omega^1(\mathbb{R}/\mathbb{Z})$ be the one-form dual to $\partial/\partial t$, i.e., $\langle \mathbf{d}t, \partial/\partial t\rangle = 1$. Direct verification

$$\beta = -\mathbf{i}_{\frac{\partial}{\partial t}}\beta \wedge \mathbf{d}t.$$

Let $\Lambda_{[u]}([s],[t]) := ([s],[t+u])$. If $\Lambda_{[u]}^{*}\beta = \beta, \forall [u] \in \mathbb{R}/\mathbb{Z} \Longrightarrow$ $\Lambda_{[u]}^{*}\left(\mathbf{i}_{\frac{\partial}{\partial t}}\beta\right) = \mathbf{i}_{\frac{\partial}{\partial t}}\beta, \forall [u] \in \mathbb{R}/\mathbb{Z} \Longrightarrow \mathbf{i}_{\frac{\partial}{\partial t}}\beta$ depends only on $[s] \Longrightarrow$

$$\int_{(\mathbb{R}/\mathbb{Z})\times(\mathbb{R}/\mathbb{Z})} \beta = -\int_{\mathbb{R}/\mathbb{Z}} \iota_1^{*}\mathbf{i}_{\frac{\partial}{\partial t}}\beta, \qquad (*)$$

$\iota_1 : (\mathbb{R}/\mathbb{Z}) \ni [s] \longmapsto ([s],[0]) \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ standard embedding.

Compute

$$T_{([s],[t])}\psi\left(a\frac{\partial}{\partial s}, b\frac{\partial}{\partial t}\right) = T_{\varphi([s])}\Phi_{[t]}\left(aT_{[s]}\varphi\left(\frac{\partial}{\partial s}\right) + b1_M(\varphi([s]))\right)$$

$$T_{([s],[t])}\Lambda_{[u]}\left(a\frac{\partial}{\partial s}, b\frac{\partial}{\partial t}\right) = \left(a\frac{\partial}{\partial s}, b\frac{\partial}{\partial t}\right)([s],[t+u]),$$

and conclude from $\psi \circ \Lambda_{[u]} = \Phi_{[u]} \circ \psi$ and $(\mathbb{R}/\mathbb{Z})$-invariance of $\alpha \in \Omega^2(M)^{\mathbb{R}/\mathbb{Z}}$ that $\Lambda_{[u]}^{*}\psi^{*}\alpha = \psi^{*}\Phi_{[u]}^{*}\alpha = \psi^{*}\alpha, \forall [u] \in \mathbb{R}/\mathbb{Z}$.

Thus we can apply formula (*) for $\beta = \psi^{*}\alpha$. LHS is what we want. Compute RHS.

$$
\begin{aligned}
\left(\iota_1^*\mathbf{i}_{\frac{\partial}{\partial t}}\psi^*\alpha\right)([s])\left(a\frac{\partial}{\partial s}\right) &= \left(\mathbf{i}_{\frac{\partial}{\partial t}}\psi^*\alpha\right)([s],[0])\left(a\frac{\partial}{\partial s},0\right)\\
&= (\psi^*\alpha)([s],[0])\left(\left(0,\frac{\partial}{\partial t}\right),\left(a\frac{\partial}{\partial s},0\right)\right)\\
&= \alpha\left(\psi([s],[0])\right)\left(T_{([s],[0])}\psi\left(0,\frac{\partial}{\partial t}\right),T_{([s],[0])}\psi\left(a\frac{\partial}{\partial s},0\right)\right)\\
&= \alpha\left(\varphi([s])\right)\left(1_M(\varphi([s])),aT_{[s]}\varphi\left(\frac{\partial}{\partial s}\right)\right)\\
&= \left(\mathbf{i}_{1_M}\alpha\right)\left(\varphi([s])\right)\left(T_{[s]}\varphi\left(a\frac{\partial}{\partial s}\right)\right)\\
&= \varphi^*\left(\mathbf{i}_{1_M}\alpha\right)([s])\left(a\frac{\partial}{\partial s}\right),
\end{aligned}
$$

i.e., $\iota_1^*\mathbf{i}_{\frac{\partial}{\partial t}}\psi^*\alpha = \varphi^*\left(\mathbf{i}_{1_M}\alpha\right)$ which proves the formula. $\square$

PROOF OF EXISTENCE OF μ

If $\mathbb{R}/\mathbb{Z}$-action does not admit a momentum map, the action is not trivial, because trivial action has 0 as a momentum map. So, assume that the action is not Hamiltonian. Then $\mathbf{i}_{1_M}\sigma$ is not exact.

Recall: X compact manifold. A **rational cohomology class** in $H^k(X;\mathbb{R})$ is real cohomology class in image of $H^k(X;\mathbb{Q}) \to H^k(X;\mathbb{R})$. Similarly, when $\mathbb{Q}$ is replaced by $\mathbb{Z}$ for integral cohomology class.

Step 1. Existence of the circle valued momentum map when $[\sigma] \in H^2(M;\mathbb{Z})$. Lemma shows that $\left[\mathbf{i}_{1_M}\sigma\right] \in H^1(M;\mathbb{Z})$. Pick a point $m_0 \in M$, let γ_m be an arbitrary smooth path connecting m_0 to m in M, and define the smooth map $\mu : M \to \mathbb{R}/\mathbb{Z}$ by

$$\mu(m) := \left[\int_{\gamma_m} \mathbf{i}_{1_M}\sigma\right].$$

μ is well defined: $\widetilde{\gamma}_m$ another path connecting m_0 to m, then $\gamma_m * (-\widetilde{\gamma}_m)$ closed loop. $\mathbf{i}_{1_M}\sigma \in H^1(M;\mathbb{Z}) \Rightarrow$ all its periods are integral, i.e., $\int_{\gamma_m*(-\widetilde{\gamma}_m)} \mathbf{i}_{1_M}\sigma =: k \in \mathbb{Z}$. Thus

$$\int_{\gamma_m} \mathbf{i}_{1_M}\sigma = \int_{\widetilde{\gamma}_m} \mathbf{i}_{1_M}\sigma + k \quad \Rightarrow \quad \left[\int_{\gamma_m} \mathbf{i}_{1_M}\sigma\right] = \left[\int_{\widetilde{\gamma}_m} \mathbf{i}_{1_M}\sigma\right].$$

$$\forall v_m \in T_mM \quad \Rightarrow \quad T_m\mu(v_m) = T_{\int_{\gamma_m}\mathbf{i}_{1_M}\sigma}\pi\Big(\mathbf{i}_{1_M}\sigma(m)(v_m)\Big)$$

$$\Rightarrow \quad (\mu^*\lambda)(m)(v_m) = \lambda(\mu(m))\left(T_m\mu(v_m)\right) = \left(\mathbf{i}_{1_M}\sigma\right)(m)(v_m),$$

by definition of λ. Thus, σ admits the circle valued momentum map μ.

Step 2. Existence of the circle valued momentum map when $[\sigma] \in H^2(M;\mathbb{Q})$. Lemma shows that $\left[\mathbf{i}_{1_M}\sigma\right] \in H^1(M;\mathbb{Q})$. Thus there is a $k \in \mathbb{N}$ such that $\left[\mathbf{i}_{1_M}(k\sigma)\right] = k\left[\mathbf{i}_{1_M}\sigma\right] \in H^1(M;\mathbb{Z})$. Since the $\mathbb{R}/\mathbb{Z}$-action clearly preserves $k\sigma$, by Step 1, the symplectic form $k\sigma$ on M admits a circle valued momentum map on M.

Step 3. Existence of the circle valued momentum map when $[\sigma] \in H^2(M;\mathbb{R})$ is irrational. Recall de Rham theorem for G-invariant forms: let G be a connected compact Lie group acting smoothly on a compact manifold X. Let $\Omega^*(X)^G$ denote the set of G-invariant forms. Then the inclusion map $i\colon \Omega^*(X)^G \to \Omega^*(X)$ induces an isomorphism $H^*(X;\mathbb{R})^G \cong H^*(X;\mathbb{R})$ in real cohomology.

Our case: $H^2(M;\mathbb{R})^{\mathbb{R}/\mathbb{Z}} \cong H^2(M;\mathbb{R})$ by the compactness of M; $m := \dim_{\mathbb{R}}\left(H^2(M;\mathbb{R})\right) = \dim_{\mathbb{Q}}\left(H^2(M;\mathbb{Q})\right)$ second Betti number.

Choose a $\mathbb{Q}$-basis of $H^2(M;\mathbb{Q})$; then it is also a $\mathbb{R}$-basis of $H^2(M;\mathbb{R}) \cong H^2(M;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ and hence $H^2(M;\mathbb{Q}) \cong \mathbb{Q}^m$ as $\mathbb{Q}$-vector spaces, $H^2(M;\mathbb{R}) \cong \mathbb{R}^m$ as $\mathbb{R}$-vector spaces. Endow $H^2(M;\mathbb{R})$ with the topology induced by this linear isomorphism $\Rightarrow H^2(M;\mathbb{Q})$ is dense in $H^2(M;\mathbb{R}) \cong H^2(M;\mathbb{R})^{\mathbb{R}/\mathbb{Z}}$.

Since $0 \neq [\sigma] \in H^2(M;\mathbb{R})^{\mathbb{R}/\mathbb{Z}}$ because σ is a symplectic form, we can complete to a basis $\{[\sigma], [\omega_1], \ldots [\omega_{m-1}]\}$ of $H^2(M;\mathbb{R})^{\mathbb{R}/\mathbb{Z}}$.

So, $\sigma, \omega_1, \ldots, \omega_{m-1} \in \Omega^2_{\text{closed}}(M)^{\mathbb{R}/\mathbb{Z}}$ are linearly independent and hence $V := \text{span}_{\mathbb{R}}\{\sigma, \omega_1, \ldots \omega_{m-1}\}$ is a m-dimensional vector subspace of $\Omega^2_{\text{closed}}(M)^{\mathbb{R}/\mathbb{Z}}$ isomorphic to $H^2(M;\mathbb{R})^{\mathbb{R}/\mathbb{Z}}$, the isomorphism being given by its values on the basis: $\sigma \mapsto [\sigma]$, $\omega_k \mapsto [\omega_k]$, for $k = 1, \ldots, m-1$. Embed by this isomorphism the $\mathbb{Q}$-vector space $H^2(M;\mathbb{Q})$ in V; its image U is a dense $\mathbb{Q}$-vector subspace of V.

Because non-degeneracy is an open condition, it follows that the set of $\mathbb{R}/\mathbb{Z}$-invariant symplectic forms in V is *open* and also *non-empty* since $\sigma \in V$. Because U is dense in V, it follows that we can find a form $\omega \in U$, hence necessarily closed and $\mathbb{R}/\mathbb{Z}$-invariant, so close to $\sigma \in V$ that it is symplectic. The problem has now been reduced to the situation studied in Step 2 with σ replaced by ω.

PROOF THAT μ IS MORSE-BOTT-NOVIKOV

This is equivalent to showing that the standard lift $\widetilde{\mu} : \widetilde{M} \ni (m,t) \mapsto t \in \mathbb{R}$ is Morse-Bott, where $\widetilde{M} = \{(m,t) \in M \times \mathbb{R} \mid \mu(m) = [t]\}$.

• Let $\omega \in \Omega^2(M)^{\mathbb{R}/\mathbb{Z}}$ be the $\mathbb{R}/\mathbb{Z}$-invariant symplectic form just constructed. Since $\widetilde{\pi} : \widetilde{M} \ni (m,t) \mapsto m \in M$ is a covering space it follows that $\widetilde{\pi}^*\omega \in \Omega^2\left(\widetilde{M}\right)$ is a symplectic form on $\widetilde{M}$.

• $\mathbb{R}/\mathbb{Z}$ acts on $\widetilde{M}$ by $\Psi_{[s]}(m,t) := (\Phi_{[s]}(m), t)$; this is in $\widetilde{M}$ because μ is $(\mathbb{R}/\mathbb{Z})$-invariant. To see this, note that it suffices to prove that $T_m\mu(1_M(m)) = 0$, which follows from the following computation:

$$\begin{aligned}
T_{\mu(m)}L_{-\mu(m)}T_m\mu(1_M(m)) &= \boldsymbol{\delta}\mu(m)(1_M(m)) \qquad \text{(def of log. der.)}\\
&= (\mu^*\lambda)(m)(1_M(m))\\
&= \left(\mathbf{i}_{1_M}\omega\right)(m)(1_M(m)) \qquad \text{(def of } \mu\text{)}\\
&= \omega(m)\left(1_M(m), 1_M(m)\right) = 0.
\end{aligned}$$

• $\widetilde{\pi} \circ \Psi_{[s]} = \Phi_{[s]} \circ \widetilde{\pi}$ and $\mathbb{R}/\mathbb{Z}$-invariance of ω implies that the $\mathbb{R}/\mathbb{Z}$-action Ψ on $\widetilde{M}$ is symplectic.

• $\widetilde{\mu} : \widetilde{M} \ni (m,t) \mapsto t \in \mathbb{R}$ is a momentum map of Ψ. Indeed, since $1_{\widetilde{M}}(m,t) = (1_M(m),(t,0))$, so $1_{\widetilde{M}}$ and 1_M are $\widetilde{\pi}$-related, for any $m \in M$, $v_m \in T_mM$, $t \in \mathbb{R}$, and $r = (\boldsymbol{\delta} f)(m)(v_m)$, we have

$$\begin{aligned}\mathbf{i}_{1_{\widetilde{M}}}(\widetilde{\pi}^*\omega)(m,t)(v_m,(t,r)) = \mathbf{i}_{1_M}\omega(m)(v_m) &= (\mu^*\lambda)(m)(v_m)\\ &= \boldsymbol{\delta}\mu(m)(v_m) = \mathbf{d}\widetilde{\mu}(m,t)(v_m,(t,r)).\end{aligned}$$

• $\widetilde{\mu} \circ \Psi_{[s]} = \widetilde{\mu}$, i.e., μ is $(\mathbb{R}/\mathbb{Z})$-invariant.

Thus, the problem is reduced to showing that the standard invariant momentum map of a circle action is Morse-Bott; well-known.

• Recall the full proof. GIVEN: (M,ω) compact symplectic, $\Phi : (\mathbb{R}/\mathbb{Z}) \times M \to M$ symplectic action with invariant momentum map $\mathbf{J} : M \to \mathbb{R}$.

First: Fixed point manifold $M^{\mathbb{R}/\mathbb{Z}}$ equals Crit($\mathbf{J}$). $\mathbb{R}/\mathbb{Z}$ is connected, so it is generated by a neighborhood of the identity; so $m \in M^{\mathbb{R}/\mathbb{Z}}$ if and only if $1_M(m) = 0$. But $\omega(m)(1_M(m), v_m) = \mathbf{dJ}(m)(v_m)$, $\forall v_m \in T_m M$, so (ω non-degenerate) $1_M(m) = 0 \Longleftrightarrow \mathbf{dJ}(m) = 0$.

Second: Computation of Hess($\mathbf{J}$). $m_0 \in F \subset$ Crit($\mathbf{J}$), $u, v \in T_{m_0}M$, and take vector fields $\widetilde{u}, \widetilde{v}$ such that $\widetilde{u}(m_0) = u$, $\widetilde{v}(m_0) = v$. Thus,

$$\begin{aligned}
\text{Hess}(\mathbf{J})(m_0) &= \boldsymbol{\pounds}_{\widetilde{u}}\left(\langle \mathbf{dJ}, \widetilde{v}\rangle\right)(m_0) = \boldsymbol{\pounds}_{\widetilde{u}}\left(\omega\left(1_M, \widetilde{v}\right)\right)(m_0) \\
&= \left(\boldsymbol{\pounds}_{\widetilde{u}}\omega\right)(m_0)\left(1_M(m_0), v\right) + \omega(m_0)\left(\left[\widetilde{u}, 1_M\right](m_0), v\right) \\
&\qquad + \omega(m_0)\left(1_M(m_0), \left[\widetilde{u}, \widetilde{v}\right](m_0)\right) \\
&= \omega(m_0)\left(v, \left[1_M, \widetilde{u}\right](m_0)\right) \\
&= \omega(m_0)\left(v, \left.\frac{d}{dt}\right|_{t=0}\left(T\Phi_{[-t]} \circ \widetilde{u} \circ \Phi_{[t]}\right)(m_0)\right) \\
&= \omega(m_0)\left(v, \left.\frac{d}{dt}\right|_{t=0} T_{m_0}\Phi_{[-t]}(u)\right).
\end{aligned}$$

However, $T_{m_0}\Phi_{[t]} : T_{m_0}M \to T_{m_0}M$ is the flow of the linearized vector field $1'_M(m_0) : T_{m_0}M \to T_{m_0}M$ and hence

$$(\mathrm{Hess}\,\mathbf{J})(m_0)(u,v) = \omega(m_0)\left(v, -1'_M(m_0)(u)\right) = \omega(m_0)\left(1'_M(m_0)(u), v\right).$$

Recall that the symplectic representation $T_{m_0}\Phi_{[t]} : (T_{m_0}M, \omega(m_0)) \to (T_{m_0}M, \omega(m_0))$ of $\mathbb{R}/\mathbb{Z}$ admits an invariant momentum map $\mathbf{L} : T_{m_0}M \to \mathbb{R}$ whose expression is

$$\mathbf{L}(v) = \frac{1}{2}\omega(m_0)\left(1'_M(m_0)(v), v\right), \qquad \forall v \in T_{m_0}M$$

and hence $(\mathrm{Hess}\,\mathbf{J})(m_0)(u,u) = 2\mathbf{L}(u)$ for all $u \in T_{m_0}M$.

Third: Each connected component F of $M^{\mathbb{R}/\mathbb{Z}} = \mathrm{Crit}(\mathbf{J})$ has even index. Obviously, if $u \in T_{m_0}F$, both the Hessian and $\mathbf{L}$ vanish. So we need to compute the Hessian on a subspace transversal to $T_{m_0}F$ in order to determine the index of F. Since $T_{m_0}F = T_{m_0}\left(M^{\mathbb{R}/\mathbb{Z}}\right) = (T_{m_0}M)^{\mathbb{R}/\mathbb{Z}}$ is a symplectic vector subspace of $(T_{m_0}M, \omega(m_0))$, its $\omega(m_0)$-orthogonal complement W is also a symplectic subspace of $(T_{m_0}M, \omega(m_0))$ and we have $T_{m_0}M = (T_{m_0}M)^{\mathbb{R}/\mathbb{Z}} \oplus W$.

So, compute $(\mathrm{Hess}\,\mathbf{J})(m_0)|_{W\times W}$. Only fixed point of the $\mathbb{R}/\mathbb{Z}$-symplectic representation on W is 0. Well-known linear algebra: $\mathbb{R}/\mathbb{Z}$-symplectic representation space $W \Rightarrow W = \oplus_{j=1}^{k} W_j$, where $\dim W_j = 2$, $\omega(m_0)$-orthogonal sum, W_j irreducible representations.

For any irreducible symplectic representation of $\mathbb{R}/\mathbb{Z}$ on a two-dimensional symplectic vector space $(U, \mathbf{d}q \wedge \mathbf{d}p)$, the associated momentum map has the expression $U \ni (q,p) \mapsto \frac{a}{2}(q^2+p^2) \in \mathbb{R}$, where $a \in \mathbb{R}$ is the weight of the representation.

So, if $w_1 + \cdots + w_k \in \oplus_{j=1}^{k} W_j$, we get

$$
\begin{aligned}
&(\mathrm{Hess}\,\mathbf{J})(m_0)(w_1 + \cdots + w_k, w_1 + \cdots + w_k) \\
&\quad = \sum_{j=1}^{k} (\mathrm{Hess}\,\mathbf{J})(m_0)(w_j, w_j) = \sum_{j=1}^{k} 2\mathbf{L}(w_j, w_j) = \sum_{j=1}^{k} a_j \left(q_j^2 + p_j^2\right),
\end{aligned}
$$

where $a_j \in \mathbb{R}$ are the weights of the irreducible $\mathbb{R}/\mathbb{Z}$-representations and (q_j, p_j) are the symplectic coordinates of $w_j \in W_j$, $j = 1, \ldots, k$. Lemma above and this formula show that $\mathbf{J}$ is Morse-Bott and that the index of the connected component $F \subset \mathrm{Crit}\,\mathbf{J}$ equals twice the number of the negative weights $a_j \in \mathbb{R}$, so is even. $\square$

EQUIVARIANT DARBOUX VIA MELBOURNE AND DELLNITZ

WARNING: There are proofs in the literature that invoke the equivariant Darboux theorem. They are incomplete, because the equivariant Darboux theorem is stated incorrectly. First time noted by Montaldi, Roberts, Stewart [1988]. This important remark is due to Melbourne and Dellnitz [1993], who correct the statements in the literature by doing much more: they give an equivariant Williamson theorem for compact representations, i.e., they give a normal form for infinitesimally symplectic matrices commuting with the symplectic representation of a compact Lie group.

G-relative Darboux: $\omega_0, \omega_1 \in \Omega^2(M)$ symplectic, no hypotheses on M. G Lie group acting properly on M and symplectically with respect to both ω_0 and ω_1. Assume $\omega_0(g \cdot m)(u, v) = \omega_1(g \cdot m)(u, v)$, $\forall g \in G$ and $u, v \in T_{g \cdot m}M$. Then there exist two G-invariant neighborhoods U_0 and U_1 of $G \cdot m$ and a G-equivariant diffeomorphism $\Psi : U_0 \to U_1$ such that $\Psi|_{G \cdot m} = Id$ and $\Psi^* \omega_1 = \omega_0$.

Now let (M, ω) symplectic and $m_0 \in M^G$. Take a neighborhood in which ω is constant by the usual Darboux theorem. Take its orbit, so get an open G-invariant set U containing m_0. Let ω_0 be the constant symplectic form on U which at m_0 equals $\omega(m_0)$; it is trivially G-invariant. Apply theorem to $\omega|_U$ and ω_0:

There is an open G-invariant neighborhood $V \subset U$ of m_0 and a G-equivariant diffeomorphism $\Psi : V \to \Psi(V) \subset U$ such that $\Psi(m_0) = m_0$ and $\Psi^* \omega = \omega_0$.

The problem is that on $T_{m_0}M$, the G-invariant symplectic forms do NOT coincide with $\mathbf{d}q^i \wedge \mathbf{d}p_i$. There are many of them and they depend on the nature of the symplectic representation: real, complex, quaternionic. In many books, the conclusion is that $\omega_0 = \mathbf{d}q^i \wedge \mathbf{d}p_i$ which is FALSE. E.g., Theorem 22.2 in Guillemin-Sternberg is false, as pointed out by Melbourne and Dellnitz. This mistake then propagates, e.g., A. Cannas da Silva's Spinger LNM [2001].

Example: Non-isomorphic symplectic forms on $\mathbb{R}^2$.

$\mathbb{R}^2 \cong \mathbb{C}$, SO(2) acts by isometries by $e^{i\theta} \cdot z := e^{i\theta} z$ and so it is symplectic relative to both symplectic forms

$$\omega_1(z,w) := \mathrm{Im}(i\bar{z}w), \qquad \omega_2(z,w) := -\mathrm{Im}(i\bar{z}w)$$

Recall that a symplectic form ω on $\mathbb{R}^2$ is SO(2)-invariant if

$$\omega(e^{i\theta}z, e^{i\theta}w) = \omega(z,w), \quad \forall \theta \in \mathbb{R}, \quad \forall z, w \in \mathbb{C}.$$

ω_1 and ω_2 are SO(2)-equivariantly symplectically isomorphic if there is an invertible linear equivariant map $P : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\omega_1(Pz, Pw) = \omega_2(z,w)$, $\forall z, w \in \mathbb{C}$. So we must have $Pz = ke^{i\theta}z$, for some fixed $\theta \in \mathbb{R}$ and $k > 0$. Therefore

$$\begin{aligned}\omega_1(Pz, Pw) &= \mathrm{Im}\left(i\overline{Pz}Pw\right) = \mathrm{Im}\left(i\overline{ke^{i\theta}z}ke^{i\theta}w\right) = \mathrm{Im}\left(ik^2e^{-i\theta}\bar{z}e^{i\theta}w\right)\\ &= k^2\,\mathrm{Im}\left(i\bar{z}w\right) = k^2\omega_1(Pz, Pw)\end{aligned}$$

This shows that ω_1 and ω_2 are never isomorphic.

ELEMENTARY REPRESENTATION THEORY

$\rho : G \to \mathrm{Aut}(V)$ a left representation of the compact Lie group G on a finite dimensional vector space V.

$$L_G(V) := \{T \in L(V) \mid \rho(g) \circ T = T \circ \rho(g), \forall g \in G\}$$

Vector subspace $U \subset V$ is **irreducible** if $G \cdot U \subset U$ and it has no proper invariant subspaces. If U is irreducible, then $L_G(V)$ is a ral division ring and hence isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We have $V = U_1 \oplus \cdots \oplus U_k$, each U_i irreducible. Sum up all those U_i that are G-isomorphic to obtain the **isotypic** decomposition $V = W_1 \oplus \cdots \oplus W_p$. The isotypic decomposition is unique and each W_j is invariant under all elements of $L_G(V)$.

Let $W = U \oplus \cdots U$ be an isotypic component, m summands of an irrep U, so $L_G(U) \cong \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, so $L_G(U) \cong L(\mathbb{F}^m) = \mathfrak{gl}(m, \mathbb{F})$. So, think of W as $\mathbb{F}^m$. Then the isotypic component $\mathbb{F}^m$ is **real, complex**, or **quaternionic** according to $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

A **symplectic form** on $\mathbb{F}^m$ is a non-degenerate bilinear skew-symmetric bilinear map $\omega : \mathbb{F}^m \times \mathbb{F}^m \to \mathbb{R}$.

Two symplectic forms ω_0 and ω_1 on $\mathbb{F}^m$ are **isomorphic over** $\mathbb{F}$ if there exists a $\mathbb{F}$-linear map $T : \mathbb{F}^m \to \mathbb{F}^m$ such that $\omega_0(Tv, Tw) = \omega_1(v, w)$. In this case T is necessarily isomorphic.

Let ω be a G-invariant symplectic form on V, $\omega_i := \omega|_{W_i \times W_i}$. Then ω_i is a G-invariant symplectic form on W_i. Any two G-invariant symplectic forms ω_0, ω_1 on V are G-isomorphic iff all ω_{0i} and ω_{1i} are G-isomorphic for each i.

W be an isotypic component of G-representation, so $L_G(V) \cong \mathbb{F}^m$, $\mathbb{F}$ a division ring. Then there is a bijective correspondence between real G-invariant symplectic forms on W and symplectic forms on $\mathbb{F}^m$. Two G-invariant real symplectic forms on W are isomorphic iff the corresponding symplectic forms on $\mathbb{F}^m$ are isomorphic over $\mathbb{F}$.

CLASSIFICATION OF SYMPLECTIC FORMS OVER REAL DIVISION RINGS

$W = \mathbb{F}^m$ isotypic component. Coordinates: $(x_1, \ldots, x_m) \in \mathbb{R}^m$, $(z_1, \ldots, z_m) \in \mathbb{C}^m$, $(w_1, \ldots, w_m) \in \mathbb{H}^m$.

ω a real symplectic form on $\mathbb{F}^m$. Then ω is isomorphic to one of the following canonical symplectic forms:

- $\mathbb{F} = \mathbb{R}$: $\sum_{j=1}^n \mathbf{d}x_j \wedge \mathbf{d}x_{j+n}$, where $m = 2n$.
- $\mathbb{F} = \mathbb{C}$: $\mathrm{Re} \sum_{j=1}^n \mathbf{d}z_j \wedge \mathbf{d}z_{j+n} + \frac{1}{2}\rho \sum_{k=2n+1}^m \mathbf{d}z_k \wedge i\mathbf{d}\bar{z}_k$, where $0 \leq n \leq m/2$, $\rho = \pm 1$.
- $\mathbb{F} = \mathbb{H}$: $\mathrm{Re} \sum_{j=1}^n \mathbf{d}w_j \wedge \mathbf{d}w_{j+n}$, if $m = 2n$
  $\mathrm{Re} \sum_{j=1}^n \mathbf{d}w_j \wedge \mathbf{d}w_{j+n} + \frac{1}{2}\mathbf{d}w_m \wedge i\mathbf{d}\bar{w}_m$, if $m = 2n + 1$

There are $m + 1$ non-isomorphic symplectic forms on each complex isotypic component of complex dimension m.

Example: $\mathbb{R}^{10} \cong \mathbb{R}^2 \times \mathbb{C}^4$. SO(2) acts symplectically by

$$e^{i\theta} \cdot (x_1, x_2, z_1, z_2, z_3, z_4) := (x_1, x_2, e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3, e^{i\theta} z_4)$$

There are two isotypic components:

- $\mathbb{R}^2$, two trivial representations of SO(2) on $\mathbb{R}$
- $\mathbb{C}^4$, four standard representations of SO(2) on $\mathbb{C}$

By the Melbourne-Dellnitz theorem, there are five canonical symplectic forms on $\mathbb{R}^{10}$: the direct sum of $\mathbf{d}x_1 \wedge \mathbf{d}x_2$ on $\mathbb{R}^2$ with one of the symplectic forms on $\mathbb{C}^4$:

$$\pm \frac{1}{2} (\mathbf{d}z_1 \wedge i\mathbf{d}\bar{z}_1 + \mathbf{d}z_2 \wedge i\mathbf{d}\bar{z}_2 + \mathbf{d}z_3 \wedge i\mathbf{d}\bar{z}_3 + \mathbf{d}z_4 \wedge i\mathbf{d}\bar{z}_4)$$

$$\mathrm{Re}\, \mathbf{d}z_1 \wedge \mathbf{d}z_2 \pm (\mathbf{d}z_3 \wedge i\mathbf{d}\bar{z}_3 + \mathbf{d}z_4 \wedge i\mathbf{d}\bar{z}_4)$$

$$\mathrm{Re}\, (\mathbf{d}z_1 \wedge \mathbf{d}z_3 + \mathbf{d}z_2 \wedge \mathbf{d}z_4)$$

DETECT HAMILTONIAN FLOWS BY FIXED POINTS

(M, ω) symplectic manifold. $(\omega, g, \mathbb{J})$ is a **compatible triple** on (M, ω) if g is a Riemannian metric and $\mathbb{J}$ is an almost complex structure (i.e., a vector bundle automorphism $\mathbb{J}\colon TM \to TM$ satisfying $\mathbb{J}^2 = -$Identity) such that $g(\cdot,\cdot) = \omega(\cdot, \mathbb{J}\cdot)$. The standard construction of a compatible triple from a symplectic form immediately extends to the G-invariant case.

L^2_ρ is the Hilbert space of square integrable functions relative the the measure ρ.

Let G be a compact Lie group acting on a symplectic manifold (M, ω) by means of symplectomorphisms. Let $(\omega, g, \mathbb{J})$ be a G-invariant compatible triple. Let λ be a measure on M such that the Radon-Nikodym derivative of λ relative to the Riemannian measure is a bounded function on M and denote by δ_λ the formal adjoint of $\mathbf{d}$ relative to the L^2_λ inner product. Prove the following result:

Assume that each L^2_λ closed one-form decomposes L^2_λ-orthogonally as a sum of the differential of a L^2_λ smooth function and a harmonic L^2_λ one-form (i.e., in $\ker \mathbf{d} \cap \ker \delta_\lambda$) and that each cohomology class of a closed one-form in L^2_λ has a unique harmonic representative. If $\mathbb{J}$ preserves harmonic one-forms and the G-action has fixed points on every connected component, then the action is Hamiltonian.
Implied by $G = S^1$; every point of compact G is on a maximal torus.

Proof: First, if true for S^1, then true for any $\mathbb{T}^k := (S^1)^k$, $k \in \mathbb{N}$ (momentum map of a product $=$ sum of the momentum maps). Second, let G be compact Lie group whose symplectic action on M has at least a fixed point. If $\xi \in \mathfrak{g}$, then $\exp \xi$ necessarily lies in a maximal torus and the restriction of the action to the torus has fixed points. Since statement true for symplectic torus actions $\Rightarrow$ this restricted action has an invariant momentum map $\Rightarrow \exists f^\xi \in C^\infty(M)$ such that $\mathbf{d}\mathbf{i}_{\xi_M}\omega = \mathbf{d}f^\xi$, $\forall \xi \in \mathfrak{g}$. Let $\{e_1, \ldots, e_r\}$ basis of $\mathfrak{g}$ and define $\mu : M \to \mathfrak{g}^*$ by $\mu^\xi := \xi^1 f^{e_1} + \cdots + \xi^r f^{e_r}$, where $\xi = \xi^1 e_1 + \cdots + \xi^r e_r$. Then $\mathbf{i}_{\xi_M}\omega = \mathbf{d}\mu^\xi \Rightarrow \mu : M \to \mathfrak{g}^*$ defined by $\langle \mu, \xi \rangle := \mu^\xi$ is a momentum map. G is compact $\Rightarrow$ can modify μ to an equivariant momentum map. $\square$

Remark: Assumption that action has fixed points is essential. S^1-action on $(\mathbb{T}^2, \mathbf{d}\theta_1 \wedge \theta_2)$ given by $e^{2i\varphi} \cdot (e^{2i\theta_1}, e^{2i\theta_2}) := (e^{2i\theta_1}, e^{2i(\theta_2+\varphi)})$ is symplectic, free, $\mathbb{T}^2$ is Kähler, hence no fixed points. If $\exists \mathbf{J} : \mathbb{T}^2 \to \mathbb{R}$, then $\ker T_{\mathbf{t}}\mathbf{J} = \mathrm{Lie}(\mathbb{T}^2)^{\omega}_{\mathbf{t}}$, $\forall \mathbf{t} \in \mathbb{T}^2$ (Reduction Lemma). Since $\mathbb{T}^2$ is compact $\Rightarrow \mathbf{J}$ has a critical point, i.e., $\ker T_{\mathbf{t}}\mathbf{J} \neq \{0\}$, impossible. So, this action does not admit a momentum map. $\Diamond$

M Kähler $\Rightarrow \mathbb{J}$ preserves the space of harmonic one-forms. M compact $\Rightarrow \exists$ Hodge decomposition relative to measure ω^n, $2n = \dim M$.

(M, ω) compact symplectic G-manifold, G compact Lie group. Assume that the the space of harmonic one-forms is invariant under $\mathbb{J}$ (true if M is Kähler). If the G-action has fixed points on every connected component of M then it is Hamiltonian. (Will prove it.)

(M, ω) complete connected Kähler $2n$ G-manifold, G compact. If $\mathbf{i}_{\xi_M}\omega \in \mathsf{L}^2_{\omega^n}$, $\forall \xi \in \mathfrak{g}$ and action has fixed points then it is Hamiltonian.

Frankel [1959]: M compact connected Kähler S^1-manifold. If symplectic S^1-action has fixed points, then it must be Hamiltonian.

McDuff [1988]: Any symplectic circle action on a compact connected symplectic 4-manifold having fixed points is Hamiltonian. But the result is false in higher dimensions: example of a compact connected symplectic 6-manifold with a symplectic circle action which has fixed points (formed by tori), but is not Hamiltonian.

Tolman-Weitsman [2000]: If S^1-action on a compact connected symplectic manifold having fixed points is semifree (i.e., free off the fixed point set), then it is Hamiltonian.

Feldman [2001]: obstruction for a symplectic S^1-action on a compact manifold to be Hamiltonian. Deduced McDuff and Tolman-Weitsman theorems. Showed that the Todd genus of a manifold admitting a symplectic circle action with isolated fixed points is equal either to 0, in which case the action is non-Hamiltonian, or to 1, in which case the action is Hamiltonian. Any symplectic circle action on a manifold with positive Todd genus is Hamiltonian. No known examples of symplectic S^1-actions on compact connected symplectic manifolds that are not Hamiltonian but have at least one isolated fixed point.

Giacobbe [2005]: Symplectic action of $\mathbb{T}^n$-torus on a $2n$ compact connected symplectic manifold with fixed points is Hamiltonian.

Ginzburg [1992]: (M, ω) symplectic G-manifold. The action is **cohomologically free** if Lie algebra homomorphism $\Lambda : \mathfrak{g} \ni \xi \mapsto [\mathbf{i}_{\xi_M}\omega] \in \mathrm{H}^1(M, \mathbb{R})$ is injective; $\mathrm{H}^1(M, \mathbb{R})$ is Abelian Lie algebra.

• If $\mathbb{T}^k$, $k \in \mathbb{N}$, acts symplectically, then $\exists 0 \leq r < k$ such that $\mathbb{T}^r$-action is cohomologically free, and $\mathbb{T}^{k-r}$-action is Hamiltonian. Λ vanishes iff $\mathbb{T}^k$-action admits a momentum map.

• G compact $\Rightarrow \exists$ finite covering $\mathbb{T}^k \times K \to G$, K semisimple compact $\Rightarrow \exists$ symplectic action of $\mathbb{T}^k \times K$ on (M, ω). K-action is Hamiltonian, since K is semisimple. Previous result applied to $\mathbb{T}^k \Rightarrow \exists$ finite covering $\mathbb{T}^r \times (\mathbb{T}^{k-r} \times K) \to G$ such that the $(\mathbb{T}^{k-r} \times K)$-action is Hamiltonian and the $\mathbb{T}^r$-action is cohomologically free. The Lie algebra of $\mathbb{T}^{k-r} \times K$ is $\ker\left(\Lambda : \xi \mapsto [\mathbf{i}_{\xi_M}\omega]\right)$.

PROOF OF THEOREM

• **Preliminaries:** (M, ω) symplectic G-manifold, G compact. Given is G-invariant compatible triple $(\omega, g, \mathbb{J})$ and λ a measure on M such that the Radon-Nikodym derivative of the Riemannian measure with respect to λ is a bounded function on M. δ_λ formal adjoint:

$$\int_M \langle\!\langle \mathbf{d}\alpha, \beta \rangle\!\rangle \, \lambda = \int_M \langle\!\langle \alpha, \delta_\lambda \beta \rangle\!\rangle \, \lambda, \qquad \forall \alpha \in \Omega^q(M), \quad \beta \in \Omega^{q+1}(M),$$

where $\langle\!\langle \, , \rangle\!\rangle$ is the inner product on forms. By assumption we have:
(i) $\alpha \in \Omega^1(M)$, $\|\alpha\|_{\mathsf{L}^2_\lambda} < \infty$, $\mathbf{d}\alpha = 0 \Rightarrow \alpha = \mathbf{d}f + \chi$ uniquely and L^2_λ-orthogonally, where $f \in C^\infty(M)$, $\mathbf{d}f \in \mathsf{L}^2_\lambda(M)$, $\mathbf{d}\chi = 0$, $\delta_\lambda \chi = 0$, $\chi \in \mathsf{L}^2_\lambda(\wedge^1 M, g) \cap \Omega^1(M)$, i.e., χ is *harmonic*. Let $\mathcal{H}$ denote the space of harmonic one-forms.
(ii) If $[\alpha] \in \mathsf{H}^1(M, \mathbb{R})$, $\|\alpha\|_{\mathsf{L}^2_\lambda} < \infty$ has a harmonic representative, it is necessarily unique.
(iii) $\mathbb{J}\mathcal{H} \subset \mathcal{H}$.

Condition **(ii)** can be reformulated as:

(ii') If $f \in \mathrm{C}^\infty(M)$, $\|\mathbf{d}f\|_{\mathrm{L}^2_\lambda} < \infty$, and $\delta_\lambda \mathbf{d}f = 0$ then f is a constant function on each connected component of M.

Proof: Suppose **(ii')** holds and let α and β be two harmonic representatives of the same cohomology class with finite L^2_λ-norm; then $\alpha - \beta = \mathbf{d}f$ for some $f \in \mathrm{C}^\infty(M)$, $\|\mathbf{d}f\|_{\mathrm{L}^2_\lambda} < \infty$. Therefore $\delta_\lambda \mathbf{d}f = \delta_\lambda(\alpha - \beta) = 0 \stackrel{(\mathbf{ii}')}{\Longrightarrow} f$ constant on each connected component of $M \Rightarrow \alpha = \beta$. Conversely, if $\|\mathbf{d}f\|_{\mathrm{L}^2_\lambda} < \infty$ and $\delta_\lambda \mathbf{d}f = 0$, then $\mathbf{d}f$ is a smooth L^2_λ harmonic one-form representing the zero cohomology class so, by **(ii)**, f is constant on each connected component of M.

- **Step 1:** Vanishing of harmonic one-forms along infinitesimal generators. Show: $\alpha \in \Omega^1(M)$ harmonic and $\|\alpha\|_{\mathrm{L}^2_\lambda} < \infty \Rightarrow \boldsymbol{\pounds}_{\xi_M}\alpha = 0$.

This is standard for δ (Killing vector fields preserve harmonic one-forms). But we have δ_λ so we give the proof.

Note: If $\varphi : M \to M$ satisfies $\varphi^* g = g$ and $\varphi^*(\lambda) = \lambda$, then

$$\varphi^*\left(\langle\langle \nu, \rho \rangle\rangle \lambda\right) = \langle\langle \varphi^*\nu, \phi^*\rho \rangle\rangle \lambda, \qquad \forall \nu, \rho \in \Omega^1(M)$$

$F_t := \Phi_{\exp(t\xi)}$ flow of ξ_M, isometry of (M,g). $\mathbf{d}\alpha = 0 \Rightarrow \mathbf{d}F_t^*\alpha = F_t^*\mathbf{d}\alpha = 0$. Show that F_t commutes with δ_λ. $\forall \beta, \gamma \in \Omega^1(M) \Rightarrow$

$$\begin{aligned}\langle \delta_\lambda F_t^*\beta, \gamma\rangle_{\mathsf{L}^2_\lambda} &= \int_M \langle\!\langle F_t^*\beta, \mathbf{d}\gamma\rangle\!\rangle\, \lambda = \int_M F_t^*\left(\langle\!\langle \beta, (F_t)_*\mathbf{d}\gamma\rangle\!\rangle\, \lambda\right) \\ &= \int_M \langle\!\langle \beta, \mathbf{d}(F_t)_*\gamma\rangle\!\rangle\, \lambda = \int_M \langle\!\langle \delta_\lambda\beta, (F_t)_*\gamma\rangle\!\rangle\, \lambda \\ &= \int_M (F_t)_*\left(\langle\!\langle F_t^*\delta_\lambda\beta, \gamma\rangle\!\rangle\, \lambda\right) = \langle F_t^*\delta_\lambda\beta, \gamma\rangle_{\mathsf{L}^2_\lambda}\end{aligned}$$

i.e., $\delta_\lambda F_t^*\beta = F_t^*\delta_\lambda\beta$. In particular, $\delta_\lambda\alpha = 0 \Rightarrow \delta_\lambda F_t^*\alpha = F_t^*\delta_\lambda\alpha = 0$.

So, α harmonic $\Rightarrow F_t^*\alpha$ is also harmonic. Thus, in $\mathsf{H}^1(M,\mathbb{R})$, we have $[F_t^*\alpha] = F_t^\star[\alpha] = [\alpha]$ since F_t is isotopic to the identity; $F_t^\star$ isomorphism induced by diffeo F_t on cohomology. But then $[F_t^*\alpha] = [\alpha] \Rightarrow F_t^*\alpha = \alpha$ since both $F_t^*\alpha$ and α are harmonic and each cohomology class has a unique harmonic representative by hypothesis **(ii)**.

Taking the t-derivative implies that $\pounds_{\xi_M}\alpha = 0$, as stated.

• **Step 2:** Using the existence of fixed points. $\alpha \in \Omega^1(M)$ harmonic and $\|\alpha\|_{\mathsf{L}^2_\lambda} < \infty \Rightarrow 0 = \boldsymbol{\pounds}_{\xi_M}\alpha = \mathbf{i}_{\xi_M}\mathbf{d}\alpha + \mathbf{d}\mathbf{i}_{\xi_M}\alpha = \mathbf{d}\mathbf{i}_{\xi_M}\alpha$, by Step 1. Thus $\alpha(\xi_M)$ is constant on each connected component of M. Since the group action has at least one fixed point on each connected component of $M \Rightarrow \alpha(\xi_M) = 0$ on M. Therefore,

$$\left\langle \xi_M^\flat, \alpha \right\rangle_{\mathsf{L}^2_\lambda} = \int_M \alpha(\xi_M)\lambda = 0, \quad \text{where} \quad \xi_M^\flat := g(\xi_M, \cdot) \in \Omega^1(M).$$

• **Step 3:** Applying the existence of a Hodge decomposition. We have $\mathbf{d}\mathbf{i}_{\xi_M}\omega = 0$ and $\|\mathbf{i}_{\xi_M}\omega\|_{\mathsf{L}^2_\lambda} < \infty$; so, by hypothesis **(i)** $\Rightarrow \mathbf{i}_{\xi_M}\omega = \mathbf{d}f^\xi + \chi^\xi$, where $f^\xi \in C^\infty(M)$, $\chi^\xi \in \Omega^1(M)$ harmonic, $\|\mathbf{d}f^\xi\|_{\mathsf{L}^2_\lambda} < \infty$, and $\|\chi^\xi\|_{\mathsf{L}^2_\lambda} < \infty$. Now show that $\chi^\xi = 0$.

Recall definition of $\mathbb{J}$ on $\Omega^1(M)$: $(\mathbb{J}\beta)(X) = \beta(\mathbb{J}X)$, $\beta \in \Omega^1(M)$, $X \in \mathfrak{X}(M)$. So, for any $Y \in \mathfrak{X}(M)$ we have

$$\begin{aligned}(\mathbf{i}_{\xi_M}\omega)(Y) &= \omega(\xi_M, Y) = -\omega(\xi_M, \mathbb{J}(\mathbb{J}Y)) = -g(\xi_M, \mathbb{J}Y) \\ &= -\xi_M^\flat(\mathbb{J}Y) = -(\mathbb{J}\xi_M^\flat)(Y) \quad \Longleftrightarrow \quad \mathbf{i}_{\xi_M}\omega = -\mathbb{J}\xi_M^\flat.\end{aligned}$$

Let $\alpha \in \Omega^1(M)$ be arbitrary harmonic, $\|\alpha\|_{\mathsf{L}^2_\lambda} < \infty$. Then

$$\left\langle \mathbf{i}_{\xi_M}\omega, \alpha \right\rangle_{\mathsf{L}^2_\lambda} = \left\langle -\mathbb{J}\xi_M^\flat, \alpha \right\rangle_{\mathsf{L}^2_\lambda} = -\int_M \left\langle\!\left\langle \mathbb{J}\xi_M^\flat, \alpha \right\rangle\!\right\rangle \lambda$$
$$= -\int_M \left\langle\!\left\langle \xi_M^\flat, \mathbb{J}\alpha \right\rangle\!\right\rangle \lambda = -\left\langle \xi_M^\flat, \mathbb{J}\alpha \right\rangle_{\mathsf{L}^2_\lambda}.$$

Hypothesis **(iii)** $\Rightarrow$ $\mathbb{J}\alpha$ is harmonic $\Rightarrow$ $\left\langle \xi_M^\flat, \mathbb{J}\alpha \right\rangle_{\mathsf{L}^2_\lambda} = 0$ by Step 2 $\Rightarrow \mathbf{i}_{\xi_M}\omega \perp_{\mathsf{L}^2_\lambda} \mathcal{H} \Rightarrow \chi^\xi = 0$ by hypothesis **(i)**. Therefore $\mathbf{i}_{\xi_M}\omega = \mathbf{d}f^\xi$, $\forall \xi \in \mathfrak{g}$ and both sides of this identity are linear in $\xi \in \mathfrak{g}$.

• **Step 3:** Construction of an equivariant momentum map. $\{e_1, \ldots, e_r\}$ basis of $\mathfrak{g}$, define $\mu : M \to \mathfrak{g}^*$ by $\mu^\xi := \xi^1 f^{e_1} + \cdots + \xi^r f^{e_r}$, where $\xi = \xi^1 e_1 + \cdots + \xi^r e_r$. Clearly, $\mathbf{i}_{\xi_M}\omega = \mathbf{d}\mu^\xi$ which proves that $\mu : M \to \mathfrak{g}^*$, defined by the requirement that its ξ-component is μ^ξ for each $\xi \in \mathfrak{g}$, is a momentum map of the G-action.

Since G is compact, one can construct out of μ an equivariant momentum map by averaging.

PROOF OF THE FIRST COROLLARY

M Kähler G-manifold, i.e., $(\omega, g, \mathbb{J})$ is G-invariant compatible triple. For Kähler: $\omega^n = n!\mu_g$, where μ_g is the g-volume form and hence $\mathsf{L}^2_{\mu_g} = \mathsf{L}^2_{\omega^n}$. Take $\lambda = \mu_g \Rightarrow \delta_\lambda = \delta$ usual codifferential defined by g.

Repeat proof. Step 1: if a cohomology class has a harmonic $\mathsf{L}^2_{\mu_g}$ representative, then it is unique. In the hypotheses of the corollary, this is implied by the weak $\mathsf{L}^2_{\mu_g}$-Hodge decomposition (holds for all complete non-compact Riemannian manifolds) and ξ_M is Killing. Step 2 is unchanged. Step 3 follows from the weak $\mathsf{L}^2_{\mu_g}$-Hodge decomposition: By hypothesis, the closed one-form $\mathbf{i}_{\xi_M}\omega \in \mathsf{L}^2_{\mu_g}$, $\forall \xi \in \mathfrak{g} \Rightarrow \mathbf{i}_{\xi_M}\omega = \mathbf{d}f^\xi + \chi^\xi$ uniquely and $\mathsf{L}^2_{\mu_g}$-orthogonally, where $f^\xi \in \mathsf{C}^\infty(M)$ and $\chi^\xi \in \Omega^1(M)$ is harmonic, $\|\mathbf{d}f^\xi\|_{\mathsf{L}^2_{\mu_g}} < \infty$, $\|\chi^\xi\|_{\mathsf{L}^2_{\mu_g}} < \infty$. As before, $\mathbf{i}_{\xi_M}\omega = \mathbb{J}\xi_M^\flat$ and for any harmonic $\alpha \in \Omega^1(M)$, $\|\alpha\|_{\mathsf{L}^2_{\mu_g}} < \infty$, we have $\left\langle \mathbf{i}_{\xi_M}\omega, \alpha \right\rangle_{\mathsf{L}^2_{\mu_g}} = -\left\langle \xi_M^\flat, \mathbf{J}\alpha \right\rangle_{\mathsf{L}^2_{\mu_g}}$. Since M is Kähler, $\mathbb{J}\alpha$ is also harmonic. Thus, by Step 2, $\left\langle \xi_M^\flat, \mathbf{J}\alpha \right\rangle_{\mathsf{L}^2_{\lambda}} = 0$, which shows that $\chi^\xi = 0$. Step 4 is unchanged.