Non-abelian quadratic Poisson brackets

From noncommutative ODE to noncommutative Algebraic Geometry and back

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Outline

1. Introduction
   - Motivations and origins
2. Integrable Systems
3. Non-Abelian Poisson Brackets
   - "Tensor" interpretations
4. Algebraic characterization
5. Summary

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Main sources of inspiration

- Integrable Systems
  - Matrix Integrable systems generalized Manakov's
  - Bihamiltonian property and Magri-Lenard schemes
- Noncommutative algebraic Poisson geometry
  - Kontsevich NC symplectic geometry
  - Le Bryun NC@geomerty
  - "Double" Poisson strucrures and AYBE

Generalized Manakov's system

- We consider ODE systems of the form

$$\frac{dx_\alpha}{dt} = F_\alpha(\mathbf{x}), \quad \mathbf{x} = (x_1, ..., x_N), \tag{1}$$

- Here x_i are $m \times m$ -matrices and F_α are (non-commutative) polynomials. There exist systems (1) integrable for any m.
- The system

$$u_t = u^2\, v - v\, u^2, \qquad v_t = 0 \tag{2}$$

is integrable by the Inverse Scattering Method for any size m of matrices u and v.

- If u - such that $u^T = -u$, and v - a constant diagonal matrix, then (2) is equivalent to the m-dimensional Euler top. This is the famous S.V. Manakov model (1976).

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Bihamiltonian structures

- Aim - to construct an integrable generalization of (2) to the case of arbitrary N using the bi-Hamiltonian approach

Definition

Two Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are compatible if

$$\{\cdot,\cdot\}_\lambda = \{\cdot,\cdot\}_1 + \lambda\{\cdot,\cdot\}_2 \tag{3}$$

is a Poisson bracket for any λ.

- If the bracket (3) is degenerate then a hierarchy of integrable Hamiltonian ODE systems can be constructed via the following result:

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Gelfand-Magri-Zakharevich Theorem-1

Theorem

Let

$$C(\lambda) = C_0 + \lambda C_1 + \lambda^2 C_2 + \cdots, \qquad \bar{C}(\lambda) = \bar{C}_0 + \lambda \bar{C}_1 + \lambda^2 \bar{C}_2 + \cdots,$$

be the Taylor expansion of any two Casimir functions for the bracket $\{\cdot,\cdot\}_\lambda$*. Then the coefficients* $C_i, \bar{C}_j$ *are pairwise commuting with respect to both brackets* $\{\cdot,\cdot\}_1$ *and* $\{\cdot,\cdot\}_2$*.*

Gelfand-Magri-Zakharevich Theorem-2

Otherwise, if, say, the bracket $\{\cdot,\cdot\}_1$ is nondegenerate, then the ratio $R = \Pi_2\Pi_1^{-1}$, (here Π_i is the Poisson tensor for $\{\cdot,\cdot\}_i$) defines a recursion operator whose spectrum provides the set of functions in involution with respect to both brackets. In this case the formula $\Pi_k = R^k\Pi_1$ gives us an infinite sequence of pairwise compatible Poisson brackets.

Origins of NA Poisson brackets

- Mikhailov, Olver and Sokolov studied an important class of Poisson brackets related to systems (1) such that the corresponding Hamiltonian operator can be expressed in terms of left and right multiplication operators given by polynomials in $x_1, \ldots, x_N$.
- Such brackets possess the following two properties:
  - they are GL_m-adjoint invariant;
  - the bracket between traces of any two matrix polynomials $P_i(x_1, \ldots, x_N), \quad i = 1, 2$ is a trace of some other matrix polynomial P_3.
- We shall call them non-abelian Poisson brackets. and we consider the compatible pairs of non-abelian Poisson brackets, where $\{\cdot, \cdot\}_1$ is linear and $\{\cdot, \cdot\}_2$ is quadratic.

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Coordinate expressions

We consider bilinear brackets of the following form:

$$\{x^{j}_{i,\alpha}, x^{j'}_{i',\beta}\} = b^{\gamma}_{\alpha,\beta} x^{j'}_{i,\gamma} \delta^{j}_{i'} - b^{\gamma}_{\beta,\alpha} x^{j}_{i',\gamma} \delta^{j'}_{i}, \tag{4}$$

and

$$\{x^{j}_{i,\alpha}, x^{j'}_{i',\beta}\} = r^{\gamma\epsilon}_{\alpha\beta} x^{j'}_{i,\gamma} x^{j}_{i',\epsilon} + a^{\gamma\epsilon}_{\alpha\beta} x^{k}_{i,\gamma} x^{j'}_{k,\epsilon} \delta^{j}_{i'} - a^{\gamma\epsilon}_{\beta\alpha} x^{k}_{i',\gamma} x^{j}_{k,\epsilon} \delta^{j'}_{i}, \tag{5}$$

where $x^{j}_{i,\alpha}$ are entries of the matrix x_α.

Poisson conditions

Theorem

1) Formula (4) defines a Poisson bracket iff

$$b^{\mu}_{\alpha\beta} b^{\sigma}_{\mu\gamma} = b^{\sigma}_{\alpha\mu} b^{\mu}_{\beta\gamma}; \tag{6}$$

2) Formula (5) defines a Poisson bracket iff the following relations hold:

$$r^{\sigma\epsilon}_{\alpha\beta} = -r^{\epsilon\sigma}_{\beta\alpha}, r^{\lambda\sigma}_{\alpha\beta} r^{\mu\nu}_{\sigma\tau} + r^{\mu\sigma}_{\beta\tau} r^{\nu\lambda}_{\sigma\alpha} + r^{\nu\sigma}_{\tau\alpha} r^{\lambda\mu}_{\sigma\beta} = 0, \tag{7}$$

$$a^{\sigma\lambda}_{\alpha\beta} a^{\mu\nu}_{\tau\sigma} = a^{\mu\sigma}_{\tau\alpha} a^{\nu\lambda}_{\sigma\beta}, \tag{8}$$

$$a^{\sigma\lambda}_{\alpha\beta} a^{\mu\nu}_{\sigma\tau} = a^{\mu\sigma}_{\alpha\beta} r^{\lambda\nu}_{\tau\sigma} + a^{\mu\nu}_{\alpha\sigma} r^{\sigma\lambda}_{\beta\tau}. \tag{9}$$

$$a^{\lambda\sigma}_{\alpha\beta} a^{\mu\nu}_{\tau\sigma} = a^{\sigma\nu}_{\alpha\beta} r^{\lambda\mu}_{\sigma\tau} + a^{\mu\nu}_{\sigma\beta} r^{\sigma\lambda}_{\tau\alpha}. \tag{10}$$

Gl_m- and trace invariancy

Theorem

Brackets of the form (4) and (5) are both invariant with respect to GL_m-action $x_\alpha \to u x_\alpha u^{-1}$, where $u \in GL_m$. Moreover, these brackets satisfy the following property: the bracket between traces of any two matrix polynomials is a trace of a matrix polynomial. Any linear (respectively quadratic) Poisson bracket satisfying these two properties has the form (4) (respectively (5)).

Warning: There are quadratic Poisson brackets that appeared in the classical version of Inverse Scattering Method ($r-$ matrix , Drinfeld-Sklyanin etc.) However, these brackets do not satisfy the properties of our Theorem.

Compatibility of linear and quadratic brackets

- A vector $\mu = (\mu_1, ..., \mu_m)$ is an admissible if for any α, β $(a^{\sigma\epsilon}_{\alpha\beta} - a^{\epsilon\sigma}_{\beta\alpha} + r^{\sigma\epsilon}_{\alpha\beta})\mu_\sigma\mu_\epsilon = 0.$
- For any admissible vector the argument shift $x_\alpha \to x_\alpha + \mu_\alpha \, \mathrm{Id}$ yields a linear Poisson bracket with

$$b^\sigma_{\alpha\beta} = (a^{\epsilon\sigma}_{\alpha\beta} + a^{\sigma\epsilon}_{\alpha\beta} + r^{\sigma\epsilon}_{\alpha\beta})\mu_\epsilon,$$

compatible with the quadratic one.

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Poisson conditions: tensor form

Let V be a linear space with a basis $v_\alpha, \ \alpha = 1, ..., m$. Define linear operators $r, \ a \in \mathrm{End}(V \otimes V)$ by

$$r v_\alpha \otimes v_\beta = r^{\sigma\epsilon}_{\alpha\beta} v_\sigma \otimes v_\epsilon, \quad a v_\alpha \otimes v_\beta = a^{\sigma\epsilon}_{\alpha\beta} v_\sigma \otimes v_\epsilon.$$

Then the identities (7)-(10) can be rewritten in the following form:

$$r^{12} = -r^{21}, \quad r^{23} r^{12} + r^{31} r^{23} + r^{12} r^{31} = 0,$$
$$a^{12} a^{31} = a^{31} a^{12}, \quad \sigma^{23} a^{13} a^{12} = a^{12} r^{23} - r^{23} a^{12},$$
$$a^{32} a^{12} = r^{13} a^{12} - a^{32} r^{13}.$$

Here all operators act in $V \otimes V \otimes V$, by σ^{ij} we mean transposition of i-th and j-th component of the tensor product and $a^{ij}, \ r^{ij}$ mean operators $a, \ r$ acting in the product of the i-th and j-th components.

Particular case: $a = 0$

There is a subclass of brackets (5) that corresponds to zero tensor a. Relations (7), (8) mean that r is a constant skew-symmetric solution of the associative Yang-Baxter equation (Aguiar, Polischshuk, Schedler...):

$$r^{12} = -r^{21}, \quad r^{23}r^{12} + r^{31}r^{23} + r^{12}r^{31} = 0.$$

Theorem

There exists one-to-one correspondence between solutions of (7), (8) up to equivalence and exact representations of anti-Frobenius algebras up to isomorphism.

Anti-Frobenius algebras

Definition

An *anti-Frobenius algebra* is an associative algebra $\mathcal{A}$ (not necessarily with unity) with a non-degenerate anti-symmetric bilinear form $(\ ,\ )$ satisfying the following relation

$$(x, yz) + (y, zx) + (z, xy) = 0 \tag{11}$$

for all $x, y, z \in \mathcal{A}$.

In other words the form $(\ ,\ )$ defines a cyclic 1-cocycle on $\mathcal{A}$.

Example

Example

- Let $\mathcal{A}$ be associative algebra of $N \times N$-matrices with zero N-th row, l - a generic element of $\mathcal{A}^*$. Then $(x, y) = l([x, y])$ - a non-degenerate anti-symmetric bilinear form satisfying (11).
- Let $(x, y) = \mathrm{trace}([x, y]\, k^T)$, where $k \in \mathcal{A}$. Put $k_{ij} = 0, i \neq j$, $k_{ii} = \mu_i$, where $i, j = 1, ..., N-1$, and $k_{iN} = 1$, $i = 1, ..., N-1$.
- The corresponding bracket (5) is given by the following tensor r:

$$r^{ii}_{Ni} = -r^{ii}_{iN} = 1, r^{ij}_{ij} = r^{ji}_{ij} = r^{ii}_{ji} = -r^{ii}_{ij} = \frac{1}{\mu_i - \mu_j}, \qquad (12)$$

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Example-contin.

Example

The bracket (12) is equivalent to

$$r^{\alpha\beta}_{\alpha\beta} = r^{\beta\alpha}_{\alpha\beta} = r^{\alpha\alpha}_{\beta\alpha} = -r^{\alpha\alpha}_{\alpha\beta} = \frac{1}{\lambda_\alpha - \lambda_\beta}, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \dots, N. \tag{13}$$

$\lambda_1, \dots, \lambda_N$ are arbitrary pairwise distinct parameters.
For $m = 1$ we have the following scalar Poisson bracket

$$\{x_\alpha, x_\beta\} = \frac{(x_\alpha - x_\beta)^2}{\lambda_\beta - \lambda_\alpha}, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, ..., N.$$

M. Van den Bergh's Double Poisson Structures-1

- Let $\mathcal{A}$ be a f.g. associative $\mathbb{C}-$algebra
- $\mathrm{Rep}_m(\mathcal{A}) := \mathrm{Hom}(\mathcal{A}, \mathrm{Mat}_m(\mathbb{C}))$,
- $Gl_m(\mathbb{C})$ acts by conjugation on $\mathrm{Mat}_m(\mathbb{C})$)
- Question (M.VdBergh): "What kind of structures we need on $\mathcal{A}$ in order $\mathrm{Rep}_m(\mathcal{A})$ and $\mathrm{Rep}_m(\mathcal{A})^{Gl_m(\mathbb{C})}$ possess a Poisson structure?"

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M. Van den Bergh's Double Poisson Structures-2

Definition

A double bracket on $\mathcal{A}$ is a bilinear map $\{\!\{-,-\}\!\} : \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$ such that:

- $\{\!\{a, b\}\!\} = -\{\!\{b, a\}\!\}^{\circ}$, where $(a \otimes b)^{\circ} = b \otimes a$;
- $\{\!\{-,-\}\!\}$: is a derivation on its second argument (wrt the outer bimodule structure on $\mathcal{A}$): $\{\!\{a, bc\}\!\} = b\{\!\{a, c\}\!\} + \{\!\{a, b\}\!\}c$;
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M. Van den Bergh's Double Poisson Structures-3

Define $\{-,-\}: \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ by
$\{-,-\} := \mu(\{\!\{a,b\}\!\}) = \{\!\{a,b\}\!\}'\{\!\{a,b\}\!\}''$

Theorem (M.VdBergh)

Let $A, \{\!\{-,-\}\!\}$ be a double Poisson algebra. Then

- $\{-,-\}$ *is a derivation in its second argument and vanishes on commutators in its first argument.*
- $\{-,-\}$ *is anti-symmetric modulo commutators.*
- $\{-,-\}$ *makes A into a left Loday algebra ($\{-,-\}$ satisfies the following version of the Jacobi identity $\{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\}$)*
- $\{-,-\}$ *makes $A/[A,A]$ into a Lie algebra.*

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Crawley-Boevey's H_0- Poisson Structures-1

- Let $\mathcal{A}$ be a f.g. associative $\mathbb{C}-$algebra, $\mathrm{Der}(\mathcal{A})$ its derivations and $HH_0(\mathcal{A}) = \mathcal{A}/[\mathcal{A},\mathcal{A}]$;
- Any $\partial \in \mathrm{Der}(\mathcal{A}$ - "descends" under the projection $p : \mathcal{A} \mapsto HH_0(\mathcal{A})$ to the map $p(\partial) : HH_0(\mathcal{A}) \mapsto HH_0(\mathcal{A})$ such that $p(\partial)(p(a)) = p(\partial(a))$;

Definition

HH_0- Poisson structure on $\mathcal{A}$ is a Lie bracket $[-,-]$ on $HH_0(\mathcal{A})$ such that the map $[p(a),-] \in \mathrm{End} HH_0(\mathcal{A})$ is induced by some derivation $\partial_a : p(\partial_a) = [p(a),-]$.

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Crawley-Boevey's H_0- Poisson Structures-2

Example

Any double Poisson structure on $\mathcal{A}$ induces an HH_0- structure on $\mathcal{A}$ via the multiplication μ:

$$[p(a), p(b)] := p(\mu(\{\!\{a, b\}\!\})).$$

Theorem (W.Crawley-Boevey)

Each HH_0- Poisson structure on $\mathcal{A}$ defines a unique Poisson structure on $\mathbb{C}[Rep_m(\mathcal{A})^{GL_m}]$ such that

$$\{tr(a), tr(b)\} = tr([p(a), p(b)]).$$

Coordinate expression

Lemma

If $\mathcal{A}, \{\!\{-,-\}\!\}$ is a double Poisson algebra then $\mathbb{C}[Rep_m(\mathcal{A})]$ is a Poisson algebra, with Poisson bracket given by

$$\{x^j_{i,\alpha}, x^l_{k,\beta}\} = \{\!\{x_\alpha, x_\beta\}\!\}'^{j}_{k}\{\!\{x_\alpha, x_\beta\}\!\}''^{l}_{i} \tag{14}$$

where by convention we write an element x of $\mathcal{A} \otimes \mathcal{A}$ as $x' \otimes x''$ (i.e. we drop the summation sign).

Example of a classification-1

Theorem

- *Let* $\mathcal{A} = \mathbb{C} < x, y >$ *. There are five non-equivalent "quadratic"* HH_0- *Poisson structures on* $\mathcal{A}$*.*
- *There are two non-equivalent "quadratic" double Poisson algebra structures on* $\mathcal{A}$*.*

Example of a classification-1

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Example of a classification-2

Theorem

This five Poisson structures are given (in terms of their Hamilton operators) by:

- **Case 1.** $r_{11}^{12} = -1,\ r_{11}^{21} = 1;$

$$\Theta_1 = \begin{pmatrix} L_x R_y - L_y R_x & 0 \\ 0 & 0 \end{pmatrix},$$

- **Case 2.** $r_{11}^{21} = 1,\ r_{11}^{12} = -1,\ a_{21}^{22} = a_{11}^{12} = -1;$

$$\Theta_2 = \begin{pmatrix} L_x R_y - L_y R_x - L_x L_y & 0 \\ -L_y L_y & 0 \end{pmatrix},$$

Example of a classification-2

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Theorem

- **Case 3.** $r^{22}_{12} = 1,\ r^{22}_{21} = -1;$

$$\Theta_3 = \begin{pmatrix} 0 & L_y R_y \\ -L_y R_y & 0 \end{pmatrix},,$$

- **Case 4.** $a^{22}_{11} = 1.$

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Theorem

- **Case 5.** $r^{22}_{12} = 1,\ r^{22}_{21} = -1,\ a^{12}_{11} = a^{22}_{21} = 1.$

$$\Theta_5 = \begin{pmatrix} L_x L_y & L_y R_y \\ -L_y R_y + L_y L_y & 0 \end{pmatrix},$$

Back to IS-1

Applying the involution

$$r^{\gamma\sigma}_{\alpha\beta} \to r^{\alpha\beta}_{\gamma\sigma}, \qquad a^{\gamma\sigma}_{\alpha\beta} \to a^{\alpha\beta}_{\gamma\sigma}. \tag{15}$$

to our first example , we get one more quadratic structure with zero tensor a:

$$r^{\alpha\beta}_{\alpha\beta} = r^{\alpha\beta}_{\beta\alpha} = r^{\beta\alpha}_{\alpha\alpha} = -r^{\alpha\beta}_{\alpha\alpha} = \frac{1}{\lambda_\alpha - \lambda_\beta}, \quad \alpha \neq \beta, \quad 1 \leq \alpha, \beta \leq N \tag{16}$$

Back to IS-2

- In this case any vector $(\mu_1, ..., \mu_N)$ is admissible.
- All entries of the matrix $\sum_1^N x_\alpha$ are Casimir functions for both quardatic Poisson bracket $\{\cdot,\cdot\}_2$ and for the corresponding linear bracket $\{\cdot,\cdot\}_1$.
- Hamiltonians of the hierarchy commuting with respect to both $\{\cdot,\cdot\}_2$ and $\{\cdot,\cdot\}_1$ are given by

$$\operatorname{tr} x_\alpha^k, \qquad \operatorname{tr} x_\alpha^k \sum_{\beta\neq\alpha} \frac{x_\beta}{\lambda_\alpha-\lambda_\beta}, \qquad k=1,2,....$$

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Back to IS-3

- The dynamical system corresponding to the simplest Hamiltonian $\operatorname{tr} x_N$ and the Poisson structure $\{\cdot,\cdot\}_2$ has the form
$$\frac{dx_\alpha}{dt} = \frac{x_N x_\alpha - x_\alpha x_N}{\lambda_N - \lambda_\alpha}, \quad \alpha = 1, \dots, N-1.$$
- There exists the following quadratic Casimir function:
$$H = \frac{1}{2}\sum_{\alpha=1}^{N} \frac{1}{\mu_\alpha} \operatorname{tr} x_\alpha^2.$$
- The non-abelian system corresponding to this Hamiltonian and the Poisson bracket $\{\cdot,\cdot\}_2$ is given by
$$\frac{dx_\alpha}{dt} = \sum_{\beta\neq\alpha} \frac{x_\alpha x_\beta^2 - x_\beta^2 x_\alpha}{(\lambda_\alpha - \lambda_\beta)\mu_\beta} + \sum_{\beta\neq\alpha} \frac{x_\beta x_\alpha^2 - x_\alpha^2 x_\beta}{(\lambda_\alpha - \lambda_\beta)\mu_\alpha}. \tag{17}$$

Back to IS-3

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Back to IS-4

The system (17) can be written in the following bi-Hamiltonian form:

$$\frac{d\mathbf{x}}{dt} = \{\mathbf{x}, \mathrm{grad}\,(\mathrm{tr}\, H)\}_2 = \{\mathbf{x}, \mathrm{grad}\,(\mathrm{tr}\, K)\}_1,$$

where

$$K = \frac{1}{3}\sum_{\alpha=1}^{N} \frac{1}{\mu_\alpha^2}\mathrm{tr}\, x_\alpha^3.$$

If $N = 2$ system (17) is equivalent to the Manakov's (2).

Summary

- There is an interesting connection between linear and quadratic Poisson structures on representations of associative algebras and some non-commutative Hamiltonian integrable systems .
- This structures are related to different types of Hochschild cocycles : cyclic and non-cyclic
- Outlook
  - Quantization of Double Poisson structures.
  - Dynamical Double Poisson structures, Elliptic Double Poisson structures...?