Lie theory for representations up to homotopy

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Overview

...based on joint work with C. Arias Abad (University of Zurich)

Plan:

1. Representations up to homotopy
2. Differentiation
3. Integration
4. Torsion

Rep. up to homotopy – *Why?*

(A Lie algebroid, G Lie groupoid, both over M)

representation of A := flat A-connection ∇ on vector bundle E

representation of G := smooth functor $\lambda : G \to \mathfrak{gl}(E)$

$\rightsquigarrow$ good Lie theory

drawback: too few objects!

e.g.:

- for $A = TM$, topological obstructions,
- no good candidate for $\mathrm{ad}(A)$ ($\mathrm{ad}(G)$) as a rep. of A (G)!

$\rightsquigarrow$ relax notion of representation...

notice:

$$\begin{aligned} \text{representation of } A \text{ on } E &\quad\leftrightarrow\quad \text{differential on } \Omega^\bullet(A, E) \\ \text{representation of } G \text{ on } E &\quad\leftrightarrow\quad \text{differential on } C^\bullet(G, E) \end{aligned}$$

Definition

representations up to homotopy

:=

same as RHS above, but allow E to be graded vector bundle

Fundamental examples of rep. up to homotopy:

- (ordinary) representations,
- flat $\mathbb{Z}$-graded connections,
- $\exists$ essentially unique and well-behaved $\mathrm{ad}(A)$ and $\mathrm{ad}(G)$ (see work by Arias Abad and Crainic)

from now on: $A = \mathrm{Lie}(G)$

differentiation $\Rightarrow$

Theorem

- $\exists$ *natural (dg-) functor* ψ

  (rep. up to homotopy of G) $\rightarrow$ *(rep. up to homotopy of A).*

- *Corresponding chain map*

  $$\psi : C^\bullet(G, E) \rightarrow \Omega^\bullet(A, E)$$

  induces isomorphism on cohomology in certain degrees.

(generalizing work of van Est, Weinstein/Xu, Crainic,...)

- implication (conjectured by Crainic/Moerdijk):

Corollary

Second deformation cohomology of Lie algebroid integrating to a proper source-2-connected Lie groupoid vanishes.

(generalizing $H^2(\mathfrak{g},\mathfrak{g}) = 0$ for $\mathfrak{g}$ semi-simple of compact type)

- for (ordinary) representations:

  G s-1- connected $\Rightarrow$ differentiation functor is surjective

  this fails for rep.s up to homotopy $\rightsquigarrow$

  how to integrate? *where* to?

consider $A = TM$

- integration functor $\int$

  $$(\text{flat connection on } E) \to (\text{representations of } \pi_1(M)),$$

  in terms of holonomies

- K. Igusa extended $\nabla \mapsto \mathrm{Hol}_\nabla$ to $\mathbb{Z}$-graded connections,

  crucial: holonomies for higher dim. simplices appear!

- flatness of $\mathbb{Z}$-graded connection $D \Rightarrow$

  $$\text{coherence equations for } \mathrm{Hol}_D,$$

  e.g.

formalization of Igusa's construction:

- replace $\pi_1(M)$ by simplicial set $\pi_\infty(M)$ with k-simplices
$$\{\sigma : \Delta_k \to M\},$$
- replace rep. (of $\pi_1(M)$) by rep. up to homotopy (of $\pi_\infty(M)$):
  rep. up to homotopy can be defined in terms of nerve NG
  $\rightsquigarrow$ def. generalizes to simplicial sets
  $\rightsquigarrow$ notion of rep. up to homotopy of $\pi_\infty(M)$
- Igusa's construction as a map
$$(\text{rep.s up to homotopy of } TM) \to (\text{rep.s up to homotopy of } \pi_\infty(M))$$

extending to morphisms and arbitrary Lie algebroids yields

Theorem

$\exists$ natural A_∞-functor of dg-categories

$$\int : (\textit{rep.s up to homotopy of } A) \to (\textit{rep. up to homotopy of } \pi_\infty(A)),$$

generalizing usual integration

$$(\textit{representations of } A) \to (\textit{representations of } G = \pi_1(A)).$$

Here $\pi_\infty(A) :=$ simplicial set with k-simplices $\{\sigma : T\Delta_k \to A\}$.

main contributions: K. Igusa, Block / Smith, Arias Abad / S.,

relying on work of: K.T. Chen, V.K.A.M. Gugenheim

Torsion – *Basics*

classical invariant in topology (distinguishes lens spaces)

focus on closed manifold M of odd dimension

real coefficients $\rightsquigarrow$ torsion comes in two flavours:

	Ray-Singer torsion	Reidemeister torsion
nature:	norm τ_1 on det $H(M)$	norm τ_2 on det $H(M)$
flavour:	analytic	combinatorial
uses:	Hodge-theory for $\Omega(M)$	Hodge-theory for $C_K^\bullet(M)$
crucial:	ζ-regularized det. of Δ	smooth triangulation K

- Theorem of Cheeger-Müller: $\tau_1 = \tau_2$.
- Def. and Theorem extend to non-trivial coefficents systems, i.e. vector bundles with flat connections.

extensions to flat $\mathbb{Z}$- or $\mathbb{Z}_2$-graded connections:

Analytic approach:

- analytic approach extended to flat $\mathbb{Z}$-graded connections early on (Quillen, Bismuth/Lott,...)
- $\mathbb{Z}_2$-graded case more subtle, Mathai/Wu (2008), e.g. H closed 3-form on $M \rightsquigarrow$ twisted cohomology $H(\Omega(M), d + H\wedge)$

Combinatorial approach:

using integration result for rep.s up to homotopy $\rightsquigarrow$

input: triangulation K and flat superconnection D on E

output: finite-dim. complex $C_K(M, E)$ computing $H(M, E)$

applying Hodge-theory to $C_K(M, E) \rightsquigarrow$

combinatorial torsion for flat superconnections

Thank you!