Unifying Hypercomplex and Holomorphic Symplectic Structures

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Devise common framework for following two notions:

1. **Hypercomplex manifold:**
   - M is smooth manifold
   - i, j, k are three endomorphisms of T_M
   - $i^2 = j^2 = k^2 = ijk = -\operatorname{id}$
   - integrability condition satisfied for each (Nijenhuis tensor vanishes)
2. **Holomorphic symplectic manifold:**
   - X = complex manifold
   - ω = holomorphic symplectic 2-form (in every holomorphic chart, $\omega = \sum_{i<j} f_{ij} dz_i \wedge dz_j$ with $f_{ij} \in \mathcal{O}_X$)

Given a smooth manifold M, consider vector bundle $T_M \oplus T_M^*$ endowed with

- nondegenerate symmetric pairing
  $\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi Y + \eta X)$
- anchor map $\quad T_M \oplus T_M^* \to T_M \quad X+\xi \mapsto X$
- Dorfman bracket
  $(X+\xi)\circ(Y+\eta) = [X,Y] + (L_X\eta - i_Y\, d\xi)$

Hitchin, Gualtieri:

- almost complex structure: $J \in \mathrm{End}(T_M \oplus T_M^*)$ such that $J^2 = -\mathrm{id}$ and $\langle Jv, w\rangle + \langle v, Jw\rangle = 0$
- integrability condition
  $0 = \mathcal{N}_J(v,w) = Jv \circ Jw - J(v\circ Jw) - J(Jv\circ w) + J^2(v\circ w)$

Examples:

1. **Complex manifold X**
   - with complex structure $j : T_X \to T_X$
   - $J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$ is a c.s. on $T_X \oplus T_X^*$
2. **Symplectic manifold (M, ω)**
   - $\omega_\flat : T_M \to T_M^*$
   - $J = \begin{pmatrix} 0 & \omega_\flat^{-1} \\ -\omega_\flat & 0 \end{pmatrix}$ is a c.s. on $T_M \oplus T_M^*$
3. **Holomorphic Poisson manifold (X, π)**
   - complex structure $j : T_X \to T_X$
   - $\pi = (\Re\pi) + i(\Im\pi)$ with $(\Im\pi)^\sharp = -j \circ (\Re\pi)^\sharp$
   - $J = \begin{pmatrix} j & (\Re\pi)^\sharp \\ 0 & -j^* \end{pmatrix}$ is a c.s. on $T_X \oplus T_X^*$

Abstract set-up: "Courant algebroid" (Courant, Liu-Weinstein-Xu)

$E \to M$ smooth vector bundle endowed with

- $\langle , \rangle$ nondegenerate symmetric pairing on fibers
- anchor $\rho : E \to T_M$ (bundle map)
  $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ (differential operator)
  related by $\langle \mathcal{D}f, x \rangle = \frac{1}{2}\rho(x)f$
- Dorfman bracket $\circ$ ($\mathbb{R}$-bilinear operation on $\Gamma(E)$)

satisfying

- $x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z)$
- $\rho(x \circ y) = [\rho(x), \rho(y)]$
- $x \circ fy = (\rho(x)f)y + f(x \circ y)$
- $x \circ y + y \circ x = 2\mathcal{D}\langle x, y \rangle$
- $\mathcal{D}f \circ x = 0$
- $\rho(x)\langle y, z \rangle = \langle x \circ y, z \rangle + \langle y, x \circ z \rangle$

Nijenhuis concomitant of $F, G \in \mathrm{End}(E)$:

$$\begin{aligned}\mathcal{N}_{F,G}(v,w) = {} & Fv \circ Gw - F(v \circ Gw) - G(Fv \circ w) + FG(v \circ w) \\ & + Gv \circ Fw - G(v \circ Fw) - F(Gv \circ w) + GF(v \circ w)\end{aligned}$$

$v, w \in \Gamma(E)$

- Usually not a tensor.
- It is a tensor if F and G are skew-symmetric and $FG + GF = \lambda\,\mathrm{id}$ (with $\lambda \in \mathbb{R}$).

- Almost hypercomplex structure on Courant algebroid:
  $I, J, K \in \mathrm{End}(E)$ skew-symmetric (w.r.t. the pairing)
  $I^2 = J^2 = K^2 = IJK = -\mathrm{id}$
- Integrability condition:
  $\mathcal{N}_{I,J}$, $\mathcal{N}_{J,K}$, $\mathcal{N}_{K,I}$, $\mathcal{N}_{I,I}$, $\mathcal{N}_{J,J}$, and $\mathcal{N}_{K,K}$ vanish

Examples:

1. **Hypercomplex manifold $(M; i, j, k)$:**

$$I = \begin{pmatrix} i & 0 \\ 0 & -i^* \end{pmatrix} \qquad J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix} \qquad K = \begin{pmatrix} k & 0 \\ 0 & -k^* \end{pmatrix}$$

2. **Holomorphic symplectic manifold (X, ω):**

$$\omega = \omega_1 - i\omega_2 \qquad (\omega_2)_\flat = (\omega_1)_\flat \circ j$$

$$I = \begin{pmatrix} 0 & (\omega_1)_\flat^{-1} \\ -(\omega_1)_\flat & 0 \end{pmatrix} \qquad J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & (\omega_2)_\flat^{-1} \\ -(\omega_2)_\flat & 0 \end{pmatrix}$$

3. **Hyper-Poisson manifold:**
   - (i, j, k) = classical hypercomplex triple on manifold M
   - $\pi_1, \pi_2, \pi_3 \in \mathfrak{X}^2(M)$
   - $\pi_2 - \sqrt{-1}\pi_3$ is holomorphic Poisson w.r.t. i
     $\pi_3 - \sqrt{-1}\pi_1$ is holomorphic Poisson w.r.t. j
     $\pi_1 - \sqrt{-1}\pi_2$ is holomorphic Poisson w.r.t. k
   - $$I = \begin{pmatrix} i & \pi_3^\sharp \\ 0 & -i^* \end{pmatrix} \qquad J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix} \qquad K = \begin{pmatrix} k & -\pi_1^\sharp \\ 0 & -k^* \end{pmatrix}$$

Hyper-Kähler

Lemma: Hyper-Poisson $\Longrightarrow [\pi_\alpha, \pi_\beta] = 0$, $\forall \alpha, \beta \in \{1, 2, 3\}$

Lemma: Every complex structure F on a Courant algebroid $E \to M$ gives rise to a Poisson bracket on the base manifold M:

$$\{f, g\} = \langle F\mathcal{D}f, \mathcal{D}g \rangle, \qquad \forall f, g \in C^\infty(M).$$

$\mathcal{N}_{F,F} = 0 \implies$ Jacobi identity

Theorem:

IF (I, J, K) is a hypercomplex triple on a Courant algebroid and π_I, π_J, π_K are the corresponding Poisson bivectors,

THEN $[\pi_\alpha, \pi_\beta] = 0 \quad \forall \alpha, \beta \in \{I, J, K\}$.

Courant algebroid $(E, \rho, \langle , \rangle, \circ)$ endowed with almost hypercomplex triple (I, J, K).

Theorem: The following assertions are equivalent:

1. all six Nijenhuis concomitants vanish identically;
2. $\mathcal{N}_{I,J} = 0$;
3. $\mathcal{N}_{I,I} = \mathcal{N}_{I,J} = 0$;
4. there exists a hypercomplex connection ∇ satisfying $\nabla I = \nabla J = \nabla K = 0$ and $T^{\nabla}(v, w) = I\mathcal{D}\langle v, Iw\rangle + J\mathcal{D}\langle v, Jw\rangle + K\mathcal{D}\langle v, Kw\rangle$.

Courant algebroid $(E, \rho, \langle , \rangle, \circ)$ endowed with almost hypercomplex triple (I, J, K).

Definition (Hypercomplex connection):

- $\Gamma(E) \times \Gamma(E) \xrightarrow{\nabla} \Gamma(E) \quad$ ($\mathbb{R}$-bilinear)
- such that $\quad \nabla_{fv} w = f \nabla_v w$
  and $\quad \nabla_v (fw) = (\rho(v) f) w + f(\nabla_v w) - \Delta_f^{I,J,K}(v, w)$
- where

$$\Delta_f^{I,J,K}(v, w) = \langle v, w \rangle \mathcal{D}f + \langle Iv, w \rangle I\mathcal{D}f + \langle Jv, w \rangle J\mathcal{D}f + \langle Kv, w \rangle K\mathcal{D}f$$

- Torsion: $T^{\nabla}(v, w) = \nabla_v w - \nabla_w v - \frac{v \circ w - w \circ v}{2}$
  Curvature: $R^{\nabla}(v, w)x = \nabla_v \nabla_w x - \nabla_w \nabla_v x - \nabla_{\frac{v \circ w - w \circ v}{2}} x$

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4. there exists a hypercomplex connection ∇ satisfying $\nabla I = \nabla J = \nabla K = 0$ and $T^{\nabla}(v,w) = I\mathcal{D}\langle v, Iw\rangle + J\mathcal{D}\langle v, Jw\rangle + K\mathcal{D}\langle v, Kw\rangle$.

- This connection is unique and given by the explicit formula $\nabla_v w = -\frac{1}{2}K\left(Jw \circ Iv - J(w \circ Iv) - I(Jw \circ v) + JI(w \circ v)\right)$.
- Isotropic and involutive subbundles of E are Lie algebroids (Dirac structures).
  IF a Dirac structure L is stable under I, J, and K, THEN the hypercomplex connection ∇ on E induces a Lie algebroid connection on L.

Examples:

1. **Hypercomplex manifold** $(M; i, j, k)$:
   - Given (the tangent distribution of a) foliation $\mathcal{F}$, we get the isotropic and involutive subbundle $\mathcal{F} \oplus \mathcal{F}^{\perp}$.
   - IF $\mathcal{F}$ is stable under i, j, and k, THEN $\mathcal{F} \oplus \mathcal{F}^{\perp}$ is stable under I, J, and K.
   - Get torsionfree connection ∇ on the Lie algebroid $\mathcal{F} \oplus \mathcal{F}^{\perp}$ such that $\nabla I = \nabla J = \nabla K = 0$.
   - Taking $\mathcal{F} = T_M$:

Corollary (Obata 1956): Given a hypercomplex triple (i, j, k) on a smooth manifold M, the formula

$$\nabla_X Y = -\tfrac{1}{2} k\big((L_{jY} i - j L_Y i)X\big)$$

defines a torsionfree connection on M such that $\nabla i = \nabla j = \nabla k = 0$.

2 Holomorphic symplectic manifold $(X;\omega)$:

- Complex Lagrangian foliation $\mathcal{L}$ yields Dirac subbundle $\mathcal{L} \oplus \mathcal{L}^{\perp}$ stable under I, J, and K.
- Get torsionfree connection ∇ on the Lie algebroid $\mathcal{L} \oplus \mathcal{L}^{\perp}$ such that $\nabla I = \nabla J = \nabla K = 0$.

Corollary: Given a holomorphic symplectic structure $\omega = \omega_1 - \sqrt{-1}\omega_2$ on a complex manifold X, the formula

$$\nabla_X Y = -\tfrac{1}{2}(\omega_2)_\flat^{-1}\big((L_{jY}(\omega_1)_\flat)X + j^*(L_Y(\omega_1)_\flat)X\big)$$

defines a torsionfree connection on the tangent bundle to the Lagrangian foliation $\mathcal{L}$.

- This connection appears in Behrend & Fantechi's recent construction of a Gerstenhaber algebra structure on the "structure sheaf" of a Lagrangian intersection in a holomorphic symplectic manifold. (motivation: Donaldson-Thomas invariants)

Extended Poisson Manifolds (introduced by Chen-S-Xu)
Motivation: semi-classical limit of algebroid stacks arising in DQ of complex manifolds.

- Consider complex mfd X, complex structure $j : T_X \to T_X$.
- Get complex structure $J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix}$ on $T_X \oplus T_X^*$.
- Eigenbundles:

$$\begin{aligned} L &= T^{1,0} \oplus (T^{0,1})^* \qquad \text{for } \lambda = +\sqrt{-1}, \\ L^* &= T^{0,1} \oplus (T^{1,0})^* \qquad \text{for } \lambda = -\sqrt{-1}. \end{aligned}$$

- $\wedge^2 L = \wedge^2 T^{1,0} \oplus (T^{1,0} \otimes (T^{0,1})^*) \oplus \wedge^2 (T^{0,1})^*$
- An extended Poisson structure on X is a section H of $\wedge^2 L$ satisfying the Maurer-Cartan equation

$$\overline{\partial} H + \frac{1}{2}[H, H] = 0.$$

Extended Symplectic Manifolds

- X = complex manifold
  $L = T^{1,0} \oplus (T^{0,1})^*$
- Choose $\Omega \in \Gamma(\wedge^2 L)$ s.t. $\Omega^\sharp \circ \overline{\Omega}^\sharp = -\operatorname{id}_L$ (nondegeneracy).
- Get almost hypercomplex triple

$$I = \Omega^\sharp + \overline{\Omega}^\sharp \qquad J = \begin{pmatrix} j & 0 \\ 0 & -j^* \end{pmatrix} \qquad K = i(\overline{\Omega}^\sharp - \Omega^\sharp).$$

- The following assertions are equivalent:
  - $d_{L^*}\Omega + [\Omega, \Omega] = 0$
  - $[\Omega, \Omega] = 0$
  - $d_{L^*}\Omega = 0$
  - (I, J, K) is integrable

The above construction is a special instance of the following

Theorem: Given a complex structure J on a Courant algebroid (with eigenbundles L and L^*) and a section Ω of $\wedge^2 L$ such that $\Omega^\sharp \circ \overline{\Omega}^\sharp = -\,\mathrm{id}_L$, set $I = \Omega^\sharp + \overline{\Omega}^\sharp$ and $K = i(\overline{\Omega}^\sharp - \Omega^\sharp)$.
The triple (I, J, K) is hypercomplex <u>iff</u> $d_{L^*}\Omega = 0$.

Philosophy: **"In the generalized context, hypercomplex = holomorphic symplectic."**

— THANK YOU —

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