ON POLY-POISSON AND POLY-DIRAC STRUCTURES

Universidad de La Laguna

Miguel Vaquero
joint work with D. Iglesias, J.C. Marrero

Universidad de La Laguna, Spain

Abstract. Poly-Poisson structures are introduced as a higher-order extension of Poisson structures. Relations with polysymplectic and Dirac structures are discussed and several examples are presented.

1. Motivation

• **A covariant definition of a symplectic structure** A **symplectic structure** on P is a skew-symmetric vector bundle isomorphism $\Omega^\flat : TP \to T^*P$ which satisfies the following integrability condition

$$\Omega^\flat([X,Y]) = \mathcal{L}_X\Omega^\flat(Y) - \mathcal{L}_Y\Omega^\flat(X) - d(\Omega^\flat(X)(Y)) \quad \text{for } X,\ Y \in \mathfrak{X}(P).$$

• **A contravariant definition of a symplectic structure** A symplectic structure is a skew-symmetric vector bundle isomorphism $\Lambda^\sharp : T^*P \to TP$ which satisfies the integrability condition

$$[\Lambda^\sharp(\alpha), \Lambda^\sharp(\beta)] = \Lambda^\sharp\left(\mathcal{L}_{\Lambda^\sharp(\alpha)}\beta - \mathcal{L}_{\Lambda^\sharp(\beta)}\alpha - d(\beta(\Lambda^\sharp(\alpha)))\right) \quad \text{for } \alpha,\ \beta \in \Lambda^1(P).$$

If $\Lambda^\sharp : T^*P \to TP$ satisfies the previous integrability condition but it is not an isomorphism we obtain a **Poisson structure** on P (for more details, see [4]).

• **A covariant definition of a polysymplectic structure** A higher order analogous of a symplectic manifold is a k-polysymplectic manifold (see [2]). A k-**polysymplectic structure** on a manifold P is a collection of closed 2-forms $(\Omega_1, \ldots, \Omega_k)$ such that $\cap_{i=1}^k \ker(\Omega_i) = \{0\}$. Equivalently, a k-polysymplectic structure on P is a skew-symmetric vector bundle monomorphism $\overrightarrow{\Omega}^\flat : TP \to (T^1_k)^*P$, $X \to \overrightarrow{\Omega}^\flat(X) = (\Omega_1{}^\flat(X), \ldots, \Omega_k{}^\flat(X))$, from TP on $(T^1_k)^*P := T^*P \oplus \overset{k)}{\ldots} \oplus T^*P$ which verifies the integrability condition

$$\overrightarrow{\Omega}^\flat([X,Y]) = \mathcal{L}_X\overrightarrow{\Omega}^\flat(Y) - \mathcal{L}_Y\overrightarrow{\Omega}^\flat(X) - d(\overrightarrow{\Omega}^\flat(X)(Y)) \quad \text{for } X,\ Y \in \mathfrak{X}(P).$$

• **A contravariant definition of a polysymplectic structure** If we take the vector subbundle $S := \overrightarrow{\Omega}^\flat(TP)$ of $(T^1_k)^*P$ and the vector bundle isomorphism $\overrightarrow{\Lambda}^\sharp := \overrightarrow{\Omega}^{\flat^{-1}} : S \subseteq (T^1_k)^*P \to TP$ then

$$[\overrightarrow{\Lambda}^\sharp(\overline{\alpha}), \overrightarrow{\Lambda}^\sharp(\overline{\beta})] = \overrightarrow{\Lambda}^\sharp\left(\mathcal{L}_{\overrightarrow{\Lambda}^\sharp(\overline{\alpha})}\overline{\beta} - \mathcal{L}_{\overrightarrow{\Lambda}^\sharp(\overline{\beta})}\overline{\alpha} - d(\overline{\beta}(\overrightarrow{\Lambda}^\sharp(\overline{\alpha})))\right) \quad \text{for } \overline{\alpha},\ \overline{\beta} \in \Gamma(S).$$

Moreover we have that if $\overline{\alpha} = (\alpha_1, \ldots, \alpha_k) \in S_x$, $x \in P$ then

1. $\alpha_i(\overrightarrow{\Lambda}^\sharp(\overline{\alpha})) = 0$.
2. $\overline{\alpha}(\overrightarrow{\Lambda}^\sharp(\overline{\beta})) = 0 \quad$ for $\overline{\beta} \in S_x \Rightarrow \overrightarrow{\Lambda}^\sharp(\overline{\alpha}) = 0$.

2. k-poly-Poisson structures: Definition, basic properties and examples

Definition 1 A k-**poly-Poisson structure** on a manifold P is a couple $(S, \overrightarrow{\Lambda}^\sharp)$, where:

1. S is a vector subbundle of $(T^1_k)^*P = T^*P \oplus \overset{k)}{\ldots} \oplus T^*P$.
2. $\overrightarrow{\Lambda}^\sharp : S \to TP$ is a vector bundle morphism which satisfies the following conditions :
   (a) $\alpha_i(\overrightarrow{\Lambda}^\sharp(\overline{\alpha})) = 0 \quad$ for $\overline{\alpha} = (\alpha_1, \ldots, \alpha_k) \in S_x$; $i \in \{1, \ldots, k\}$ for $x \in P$.
   (b) If $\overline{\alpha}(\overrightarrow{\Lambda}^\sharp(\overline{\beta})) = 0 \quad$ for $\overline{\beta} \in S_x \Rightarrow \overrightarrow{\Lambda}^\sharp(\overline{\alpha}) = 0$ for $x \in P$.
   (c) If $\overline{\alpha}, \overline{\beta} \in \Gamma(S)$ we have that the following integrability condition holds

$$[\overrightarrow{\Lambda}^\sharp(\overline{\alpha}), \overrightarrow{\Lambda}^\sharp(\overline{\beta})] = \overrightarrow{\Lambda}^\sharp\left(\mathcal{L}_{\overrightarrow{\Lambda}^\sharp(\overline{\alpha})}\overline{\beta} - \mathcal{L}_{\overrightarrow{\Lambda}^\sharp(\overline{\beta})}\overline{\alpha} - d(\overline{\beta}(\overrightarrow{\Lambda}^\sharp(\overline{\alpha})))\right). \tag{1}$$

The k-poly-Poisson structure is said to be **regular** if the vector bundle morphism $\overrightarrow{\Lambda}^\sharp : S \subseteq (T^1_k)^*P \to TP$ has constant rank.

Example 2.1 (Poisson manifolds) Let $(S, \overrightarrow{\Lambda}^\sharp)$ be a 1-poly-Poisson structure on a manifold P. Then, it is easy to prove that the morphism $\overrightarrow{\Lambda}^\sharp : S \subseteq T^*P \to TP$ may be extended to a vector bundle morphism $\overrightarrow{\Lambda}^\sharp : T^*P \to TP$. Therefore 1-poly-Poisson structures are just Poisson structures. ⋄

Example 2.2 (Polysymplectic manifolds) It is clear that a k-polysymplectic manifold is a k-poly-Poisson manifold. In particular, if M is a smooth manifold then $(T^1_k)^*M = T^*M \oplus \overset{k)}{\ldots} \oplus T^*M$ is a k-polysymplectic manifold. In fact $(\Omega_1 = p_1^*(\Omega), \ldots, \Omega_k = p_k^*(\Omega))$ is a k-polysymplectic strucutre on $(T^1_k)^*M$ where $p_i : (T^1_k)^*M \to T^*M$ is the canonical projection over the i-th factor and Ω is the canonical symplectic structure of T^*M. ⋄

Let $(S, \overrightarrow{\Lambda}^\sharp)$ be a k-poly-Poisson structure on a manifold P. Then, we may consider the generalized distribution D on P given by $x \in P \to D(x) := \overrightarrow{\Lambda}^\sharp(S_x) \subseteq T_xP$. D is integrable and, therefore, D induces a generalized foliation on P. On the other hand, using the contravariant description of a k-polysymplectic structure of P, it is easy to prove that the k-poly-Poisson structure of P induces a k-polysymplectic structure on the leaves of D. In conclusion, we have the following result.

Theorem 2.1 *[3] A k-poly-Poisson manifold P admits a k-polysymplectic generalized foliation D. In fact, if $(S, \overrightarrow{\Lambda}^\sharp)$ is the k-poly-Poisson structure on P, then the characteristic space of D at the point $x \in P$ is $D(x) = \overrightarrow{\Lambda}^\sharp(S_x)$. D is said to be the **canonical k-polysymplectic foliation** of P*

Example 2.3 (k-poly-Poisson structures of Dirac type) Let $(\Omega_1, \ldots, \Omega_k)$ be a k-polysymplectic structure on a manifold P and $\mathcal{F}$ be a regular foliation on P such that

$$\mathcal{F}(x) \cap \mathcal{F}^\perp(x) = \{0\} \quad \text{for } x \in P$$

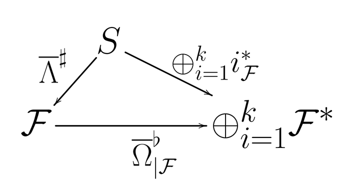
where $\mathcal{F}^\perp(x) := \cap_{i=1}^k (\Omega_i^\flat)^{-1}(\mathcal{F}(x))^\circ$. Here, $\Omega_i^\flat : TP \to T^*P$ is the vector bundle morphism induced by the 2-form Ω_i.

Now, let S be the vector subbundle of $(T^1_k)^*P$ whose characteristic space at the point $x \in P$ is

$$S_x = \{(\alpha_1, \ldots, \alpha_k) \in (T^1_k)^*P | \ (\alpha_{1|_{\mathcal{F}(x)}}, \ldots, \alpha_{k|_{\mathcal{F}(x)}}) \in \overrightarrow{\Omega}^\flat_{|\mathcal{F}(x)}(\mathcal{F}(x))\},$$

$\overrightarrow{\Omega}^\flat_{|\mathcal{F}} : \mathcal{F} \to (\mathcal{F}^*)^k$ being the vector bundle monomorphism induced by the k 2-forms $\Omega_1, \ldots, \Omega_k$ restricted to $\mathcal{F}$.

Then, one may prove that there exists a vector bundle morphism $\overrightarrow{\Lambda}^\sharp : S \subseteq (T^1_k)^*P \to TP$ such that the following diagram



is commutative. Here, $i_{\mathcal{F}} : \mathcal{F} \to TP$ is the canonical inclusion and $i_{\mathcal{F}}^*$ is the dual morphism. ⋄

Proposition 2.2 *The couple $(S, \overrightarrow{\Lambda}^\sharp)$ is a k-poly-Poisson structure on P and the canonical k-polysymplectic foliation of P is just $\mathcal{F}$.*

Remark 2.1 If $k = 1$ then the corresponding Poisson structure on P is said to be of Dirac type (see [4]).

3. k-poly-Poisson manifolds and Dirac structures

It is well-known that a Dirac structure on a manifold P is a Lagrangian subbundle L of $TP \oplus T^*P$ with respect to the canonical symmetric pairing $\langle \cdot, \cdot \rangle_+$, whose space of sections $\Gamma(L)$ is closed under the Courant bracket.

$$[(X,\alpha)(Y,\beta)] = ([X,Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + d\{1/2(\alpha(Y) - \beta(X))\} \quad \text{for } X,\ Y \in \mathfrak{X}(P) \text{ and } \alpha,\ \beta \in \Lambda^1(P).$$

Remark 3.1 A Poisson strucure Λ is a Dirac structure such that $L = graph(\Lambda^\sharp)$. Note that in this case $L \cap TP = \{0\}$.

If L is a Dirac structure on P then one may consider a generalized distribution C on P whose characteristic space at the point $x \in P$ is $C(x) = \rho(L_x)$, $\rho : L \to TP$ being the projection on the first factor. C is a presymplectic generalized foliation on P (for more details, see [1]).

Let $(S, \overrightarrow{\Lambda}^\sharp)$ be a regular k-poly-Poisson structure on P and D be the canonical k-polysymplectic foliation of P. For each $i \in \{1, \ldots, k\}$ we define the vector subbundle $L_i^{(S,\overrightarrow{\Lambda}^\sharp)}$ of $TP \oplus T^*P$ whose characteristic space at the point $x \in P$ is

$$L_i^{(S,\overrightarrow{\Lambda}^\sharp)}(x) := \{(\overrightarrow{\Lambda}^\sharp(\overline{\alpha}), \alpha_i + \gamma) \in T_xP \oplus T_x^*P | \ \overline{\alpha} = (\alpha_1, \ldots, \alpha_k) \in S_x \text{ and } \gamma \in (\overrightarrow{\Lambda}^\sharp(S_x))^\circ\}.$$

One may prove that $L_i^{(S,\overrightarrow{\Lambda}^\sharp)}$ is a Dirac structure on P.

Proposition 3.1 *[3] For each $i \in \{1, \ldots, k\}$, $L_i^{(S,\overrightarrow{\Lambda}^\sharp)}$ is a Dirac structure on P and the following properties hold*

1. $\rho(L_i^{(S,\overrightarrow{\Lambda}^\sharp)}) = D, \quad$ for $i \in \{1, \ldots, k\}$.
2. $\cap_{i=1}^k L_i^{(S,\overrightarrow{\Lambda}^\sharp)} \cap TP = \{0\}$.

Definition 2 A collection of Dirac structures $(L_1, \ldots, L_k)$ on a manifold P is said to be a k-**poly-Dirac structure** if

1. $\rho(L_i) = \rho(L_j), \quad$ for $i,\ j \in \{1, \ldots, k\}$.
2. $\cap_{i=1}^k L_i \cap TP = \{0\}$.

Corollary 3.2 *A regular k-poly-Poisson structure induces a k-poly-Dirac structure on P.*

4. Poly-Poisson structures and reduction of polysymplectic structures

We prove that, under certain regularity conditions, the space of orbits of a polysymplectic action of a Lie group on a polysymplectic manifold is a poly-Poisson manifold.

Theorem 4.1 *[3] Let $(P, \Omega_1, \ldots, \Omega_k)$ be a polysymplectic manifold and G be a Lie group that acts over P $(\Phi : G \times P \to P)$ freely, properly and satisfies $\Phi_g^*\Omega_i = \Omega_i$, $\forall g \in G$. Suppose that $V\pi$ is the vertical bundle of the principal bundle projection $\pi : P \to P/G$ and that the following conditions hold:*

1. $Im(\overrightarrow{\Omega}^\flat) \cap (V\pi)^{\circ k}$ is a subbundle, where $(V\pi)^{\circ k} = (V\pi)^\circ \times \overset{k)}{\ldots} \times (V\pi)^\circ$.

2. $\overrightarrow{\Omega}^{\flat^{-1}}(((V\pi)^{\perp\circ})^k \bigcap (V\pi)^{\circ k} \bigcap Im(\overrightarrow{\Omega}^\flat)) \subset V\pi$.

Then, there is a natural k-poly-Poisson structure on P/G.

Example 4.1 ($(T^1_k)^*Q/G$) Let Q be a manifold and G be a Lie group that acts freely and properly in Q by $\Phi : G \times Q \to Q$. We can define the $(T^1_k)^*$-lifted action in analogy with the contangent lifted action.

$$\begin{aligned}\Phi^{(T^1_k)^*} : \quad G \times (T^1_k)^*Q &\to (T^1_k)^*Q \\ (g, (\alpha_1, \ldots, \alpha_k)) &\to (\Phi^*_{g^{-1}}\alpha_1, \ldots, \Phi^*_{g^{-1}}\alpha_k).\end{aligned}$$

It can be seen that the conditions of the last theorem are satisfied and so, $(T^1_k)^*Q/G$ admits a natural poly-Poisson structure (see [3]). ⋄

5. Future work

The results in this poster will be applied to the study of classical field theories. In this direction, following the same patterns as in this poster, a Poisson extension of the notion of a multisymplectic structure may be introduced. This notion will play an important role in the reduction of classical field theories (see [3])

References

[1] T. Courant: Dirac manifolds. Trans. Amer. Math. Soc. 319 (1990).

[2] C. Gunther: The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: The local case, J. Differential Geom. 25 (1987), 23-53.

[3] D. Iglesias, J.C. Marrero, M. Vaquero: Work in progress.

[4] I.Vaisman: Lectures on the geometry of Poisson manifolds. Progress in Mathematics, 118. Birkhuser Verlag, Basel, 1994.

Acknowledgments: This work has been partially supported by MICINN (Spain) grants MTM2009-13383, MTM2009-08166-E. M.V. wishes to thank ULL (Departamento de Matemática Fundamental, Universidad de La Laguna, Spain) for the financial support and its hospitality.