# Simultaneous deformations of Maurer-Cartan elements

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# Before we start

## Example (Poisson structures)

Let $M$ be a manifold.

A Poisson structure is $\pi \in \Gamma(\wedge^2 TM)$ such that $[\pi, \pi] = 0$.

A deformation of $\pi$ is given by $\tilde{\pi} \in \Gamma(\wedge^2 TM)$ such that

$$0 = [\pi + \tilde{\pi}, \pi + \tilde{\pi}] = 2[\pi, \tilde{\pi}] + [\tilde{\pi}, \tilde{\pi}] = 2( \underbrace{d_\pi \tilde{\pi} + \frac{1}{2}[\tilde{\pi}, \tilde{\pi}]}_{\text{Maurer-Cartan equation}} ).$$

There are many mathematical structures whose deformations are given by a Maurer-Cartan equation.

In this talk we deform two structures simultaneously.

# Outline

# Example: Lie algebra morphisms

Let $U, V$ be Lie algebras.
Let $\phi\colon U \to V$ be a morphism, i.e. $d_U\phi + \frac{1}{2}[\phi,\phi]_V = 0$.

1) Deformation of morphisms:
   Let $\tilde{\phi}\colon U \to V$ be a linear map.

$$\phi + \tilde{\phi} \text{ is a morphism } \Leftrightarrow \quad \underbrace{d_{U,\phi}\tilde{\phi} + \frac{1}{2}[\tilde{\phi},\tilde{\phi}]_V = 0}_{\text{is a Maurer-Cartan equation (1966)}} \quad .$$

2) Deformation of Lie algebras and of morphisms:
   Let $\tilde{d_U} \in \wedge^2 U^* \otimes U$, $\tilde{d_V} \in \wedge^2 V^* \otimes V$.

$$\begin{cases} (U, d_U + \tilde{d_U}) \text{ and } (V, d_V + \tilde{d_V}) \text{ are Lie algebras} \\ \phi + \tilde{\phi} \text{ is a morphism between them} \end{cases}$$

$$\Leftrightarrow \underbrace{(\text{some cubic equation in } \tilde{d_U}, \tilde{d_V}, \tilde{\phi})}_{\text{is a Maurer-Cartan equation (2008)}} .$$

AIM: Find a criteria to determine when:

pair of algebraic/geometric structures

$$\Downarrow ?$$

their simultaneous deformations are given by the MC equation of some $L_\infty$-algebra.

WHY?

- $\{$deformations$\}$ acquires a natural equivalence relation.
- the deformations of $A$ and $B$ are governed by quasi-isomorphic $L_\infty$-algebras
  $\Rightarrow$ the deformation theories for $A$ and $B$ are equivalent.
- $H^1 = 0$
  $\Rightarrow$ first order deformations can be extended to (formal) deformations.

# $L_\infty$-algebras and Maurer-Cartan elements

## Definition

A $L_\infty[1]$-algebra consists of a graded vector space $W = \oplus_{i\in\mathbb{Z}} W_i$ and

$$\{\cdot,\cdots,\cdot\}_n : \otimes^n W \longrightarrow W \qquad (n \geq 1)$$

graded symmetric, of degree $1$, satisfying "higher Jacobi identities":

- $d^2 = 0$, where $d := \{\cdot\}_1$
- $d\{a,b\}_2 = \{da,b\}_2 + (-1)^{|a|}\{b,da\}_2$
- $\{\{a,b\}_2,c\}_2 \pm c.p. = \pm d\{a,b,c\}_3 \pm (\{da,b,c\}_3 \pm c.p.)$
- ...

## Definition

A Maurer-Cartan element of a $L_\infty[1]$-algebra $W$ is an element $Q \in W_0$ satisfying

$$\sum_{n=1}^{\infty} \frac{1}{n!}\{Q,\ldots,Q\}_n = 0.$$

## Example (DGLA)

Suppose that only $d := \{\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are non-zero.
Then $W[-1]$ is a differential graded Lie algebra (DGLA).
The MC equation reads

$$dQ + \frac{1}{2}\{Q,Q\}_2 = 0.$$

# Voronov's construction of $L_\infty[1]$-algebras

## Definition

A V-data consists of a quadruple $(L, \mathfrak{a}, P, \Delta)$ where

- $(L, [\cdot,\cdot])$ is a graded Lie algebra
- $\mathfrak{a}$ an abelian Lie subalgebra
- $P\colon L \to \mathfrak{a}$ a projection whose kernel is a Lie subalgebra of $L$
- $\Delta \in Ker(P)_1$ such that $[\Delta, \Delta] = 0$.

## Theorem (Th. Voronov)

*Let $(L, \mathfrak{a}, P, \Delta)$ be a V-data.*

a) $\mathfrak{a}$ *has an induced $L_\infty[1]$-structure with multibrackets ($n \geq 1$)*

$$\{a_1, \dots, a_n\} = P[\dots[[\Delta, a_1], a_2], \dots, a_n].$$

*Notation:* $\mathfrak{a}^P_\Delta$.

b) $L[1] \oplus \mathfrak{a}$ *has an induced $L_\infty[1]$-structure extending* $\mathfrak{a}^P_\Delta$.
*Notation:* $(L[1] \oplus \mathfrak{a})^P_\Delta$.

# An algebraic theorem: deformations of MC elements

## Theorem (Frégier-Z.)

*Let*

- $(L, \mathfrak{a}, P, \Delta)$ *be a V-data*
- $\Phi \in MC(\mathfrak{a}^P_\Delta)$,

*denote* $P_\Phi := P \circ e^{[\cdot, \Phi]} : L \to \mathfrak{a}$.

1) *For any* $\tilde{\Phi} \in \mathfrak{a}_0$*:*

$$\Phi + \tilde{\Phi} \in MC(\mathfrak{a}^P_\Delta) \quad \Leftrightarrow \quad \tilde{\Phi} \in MC(\mathfrak{a}^{P_\Phi}_\Delta).$$

2) *For all* $\tilde{\Delta} \in (ker(P))_1$ *and* $\tilde{\Phi} \in \mathfrak{a}_0$*:*

$$\begin{cases} [\Delta + \tilde{\Delta}, \Delta + \tilde{\Delta}] = 0 \\ \Phi + \tilde{\Phi} \in MC(\mathfrak{a}^P_{\Delta + \tilde{\Delta}}) \end{cases} \Leftrightarrow (\tilde{\Delta}, \tilde{\Phi}) \in MC((L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta).$$

# Applying the theorem

Let $U, V$ be Lie algebras. Choose

- $L = \wedge(U^* \times V^*) \otimes (U \times V)$
- $\mathfrak{a} = \wedge U^* \otimes V$
- $P\colon L \to \mathfrak{a}$ the natural projection
- $\Delta = d_U + d_V$

MC elements of $\mathfrak{a}^P_\Delta$ are exactly morphisms $U \to V$!

- $\Phi \in MC(\mathfrak{a}^P_\Delta)$.

$\Rightarrow$ 1) *For any $\tilde{\Phi}\colon U \to V$ linear:*

$$\Phi + \tilde{\Phi} \text{ is a morphism} \quad \Leftrightarrow \quad \tilde{\Phi} \in MC(\mathfrak{a}^{P_\Phi}_\Delta).$$

2) *For all $\tilde{d_U} \in \wedge^2 U^* \otimes U$, $\tilde{d_V} \in \wedge^2 V^* \otimes V$:*

$$\begin{cases} (U, d_U + \tilde{d_U}) \text{ and } (V, d_V + \tilde{d_V}) \text{ are Lie algebras} \\ \Phi + \tilde{\Phi} \text{ is a morphism between them} \end{cases}$$

$$\Leftrightarrow (\tilde{d_U} + \tilde{d_V}, \tilde{\Phi}) \in MC((L[1] \oplus \mathfrak{a})^{P_\Phi}_\Delta).$$

# Twisted Poisson structures

Let $M$ be a manifold.

## Definition

Let

- $H \in \Omega^3(M)$ be a closed 3-form
- $\pi \in \chi^2(M)$ be a bivector field.

We say

$$\pi \text{ is a } H\text{-twisted Poisson structure} \Leftrightarrow [\pi, \pi]_{Schouten} = 2(\wedge^3 \pi^\sharp)H,$$

where $\pi^\sharp \colon T^*M \to TM, \xi \mapsto \iota_\xi \pi$.

Such structures are interesting

- in geometry (Courant algebroids, Dirac structures...)
- in physics (String theory, sigma models with Wess-Zumino-Witten terms...).

# Applying the theorem

Let $M$ be a manifold. Choose

- $L = \mathcal{C}(T^*[2]T^*[1]M)[2]$
- $\mathfrak{a} = \mathcal{C}(T^*[1]M)[2]$
- $P\colon L \to \mathfrak{a}$ the restriction to the zero section
- $\Delta =$ “de Rham”

MC elements of $\mathfrak{a}^P_\Delta$ are exactly Poisson bivector fields

- $0 \in MC(\mathfrak{a}^P_\Delta)$.

$\Rightarrow$ 2) *For all* $H \in \Omega^3(M)$, *for all* $\pi \in \chi^2(M)$*:*

$$\begin{cases} dH = 0 \\ \pi \text{ is a } H\text{-twisted Poisson structure} \end{cases}$$

$$\Leftrightarrow (-H, \pi) \in MC((L[1] \oplus \mathfrak{a})^P_\Delta).$$

# The relevant subalgebra of $(L[1] \oplus \mathfrak{a})^P_\Delta$.

## Corollary

*Let $M$ be a manifold. There is an $L_\infty[1]$-algebra structure on*

$$\mathfrak{L} := \oplus_{i \geq -2}[\Omega^{i+3}(M) \oplus \chi^{i+2}(M)]$$

*whose only non-vanishing multibrackets are*

a) $d_{DeRham}$ *on differential forms*

b) $\pm[\cdot,\cdot]_{Schouten}$ *on multivector fields*

c) *for all* $n \geq 1$

$$\{H, \pi_1, \ldots, \pi_n\} \; = \; \pm(\pi_1^\sharp \wedge \cdots \wedge \pi_n^\sharp)H \in \chi^\bullet(M)$$

*where* $H \in \Omega^n(M)$ *and* $\pi_i \in \chi^\bullet(M)$*.*

*Its MC elements are exactly*

$$\{(-H, \pi) : \pi \text{ is a } H\text{-twisted Poisson structure}\}.$$

Without the algebraic theorem, it would have been hard to find an $L_\infty$-algebra as above!

# Equivalences of MC elements

Given an $L_\infty[1]$-algebra $W$, there is a map

$$W_{-1} \to \{\text{vector fields on } W_0\}, \qquad z \mapsto \mathcal{Y}^z$$

where the value of $\mathcal{Y}^z$ at $m \in W_0$ is

$$\mathcal{Y}^z|_m := dz + \{z, m\} + \frac{1}{2!}\{z, m, m\} + \frac{1}{3!}\{z, m, m, m\} + \cdots \in W_0 = T_m W_0.$$

This gives an involutive (singular) distribution on $MC(W) \subset W_0$
$\rightsquigarrow$ equivalence relation on $MC(W)$.

## Example

Let $\mathfrak{L}$ be the $L_\infty[1]$-algebra whose MC elements are twisted Poisson structures. The following coincide:

- The equivalence classes in $MC(\mathfrak{L})$
- the orbits of the (partial) group action

$$\Omega^2(M) \rtimes \mathrm{Diff}(M) \quad \circlearrowleft \quad MC(\mathfrak{L}) \subset \Omega^3(M) \times \chi^2(M)$$

$$(B, \phi) \cdot (H, \pi) = ((\phi^{-1})^* H + dB \ , \ e^B \phi_* \pi).$$

Thank you!