Applied Numerical Mathematics 62 (2012) 1718–1739

Contents lists available at SciVerse ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Supraconvergence and supercloseness in Volterra equations

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ARTICLE INFO

Article history:
Received 25 May 2011
Received in revised form 20 February 2012
Accepted 17 June 2012
Available online 28 June 2012

Keywords:
Integro-differential equations
Finite difference methods
Piecewise linear finite element methods
Supraconvergence
Supercloseness

ABSTRACT

Integro-differential equations of Volterra type arise, naturally, in many applications such as for instance heat conduction in materials with memory, diffusion in polymers and diffusion in porous media. The aim of this paper is to study a finite difference discretization of the mentioned integro-differential equations. Second convergence order with respect to the H^1 norm is established which means that the discretization proposed is supraconvergent in finite difference methods language. As the finite difference method can be seen as a piecewise linear finite element method combined with special quadrature formulas, our result establishes the supercloseness of the gradient in the finite element language. Numerical results illustrating the discussed theoretical results are included.

1. Introduction

We consider discretizations of the integro-differential equation

$$\frac{\partial u}{\partial t}(t) + Au(t) = \int_0^t B(s,t)u(s)\,ds + f(t), \quad t \in (0,T], \tag{1}$$

subject to the Dirichlet boundary condition

$$u(t) = \psi(t) \quad \text{on } \partial\Omega \times (0,T], \tag{2}$$

and with the initial condition

$$u(0) = u_0. \tag{3}$$

In (1) $u(t)$ denotes a function defined on $\overline{\Omega} \times [0,T]$ when t is fixed, Ω is a simple polygonal domain of $\mathbb{R}^2$, A and $B(s,t)$ represent the following differential operators

$$Au(t) = -\nabla.\big(\mathcal{A}\nabla u(t)\big) + \nabla.\big(\mathcal{A}_0 u(t)\big) + a_0 u(t),$$
$$B(s,t)u(t) = -\nabla.\big(\mathcal{B}(s,t)\nabla u(t)\big) + \nabla.\big(\mathcal{B}_0(s,t)u(t)\big) + b_0(s,t)u(t),$$

where $\mathcal{A}, \mathcal{A}_0, a_0$ dependent on (x,y), $\mathcal{A}_0 = [a_i]$, $\mathcal{A} = [a_{ij}]$, $i,j = 1,2$, and $a_{12} = a_{21} = a_m$. $\mathcal{B}, \mathcal{B}_0, b_0$ dependent on (x,y), s and t, $\mathcal{B}_0 = [b_i]$, $\mathcal{B} = [b_{ij}]$, $i,j = 1,2$, and $b_{12} = b_{21} = b_m$.

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0168-9274/$36.00
http://dx.doi.org/10.1016/j.apnum.2012.06.028

Integro-differential equations of type (1) arise in many applications of different branches of engineering sciences as for instance in heat conduction in materials with memory [29], in diffusion processes in porous media [10,27,36] and in diffusion in polymers [25]. In this last application, the integro-differential equation (1), with

$$\mathcal{A}_0 = \mathcal{B}_0 = 0, \qquad a_0 = b_0 = 0, \qquad \mathcal{B}(s,t) = K(t-s)\mathcal{B} \quad \text{and} \quad K(s) = \frac{1}{\tau} e^{-\frac{s}{\tau}}, \tag{4}$$

is used to model a diffusion process occurring in a swellable polymeric matrix. In this case the mass flux is assumed to be split into the sum of two mass fluxes: $J = J_F + J_{NF}$, where J_F is the mass flux given by Fick's law

$$J_F(t) = -\mathcal{A}\nabla u(t)$$

and J_{NF} satisfies the differential equation

$$\frac{\partial J_{NF}}{\partial t} + \frac{1}{\tau} J_{NF} = \frac{1}{\tau}\mathcal{B}\nabla u(t),$$

where $\tau > 0$ is a relaxation parameter (see also [34]). Eq. (1) is then established taking $J_{NF}(0) = 0$ and the mass conservation law $\frac{\partial u}{\partial t}(t) + \nabla . J(t) = f(t)$. The same equation can be used to model diffusion processes through glassy polymers. In this case the Fickian flux J_F is modified to incorporate the stress effect which is linked with the strain by the Maxwell model [12–14,39].

The development of efficient and accurate numerical methods to solve the initial boundary value problem (IBVP) defined by (1) has attracted the attention of several researchers during the last two decades. A significative number of contributions can be found in the literature. Without be exhaustive we mention [32,33,44,48] for the study of finite element semi-discrete approximations. Generally, in these papers, it is shown that several results known for finite element semi-discrete approximations for solutions of parabolic problems also hold for the corresponding semi-discrete approximations for the solutions of (1). For instance, it is established, under convenient assumptions on the partition of the domain, that piecewise linear finite element semi-discrete approximations are second order convergent with respect to the L^2-norm and they are first order convergent with respect to the H^1-norm. Similar convergent results were also established in [37] for semi-discrete lumped mass approximations with respect to discrete norms and assuming that the solutions of the continuous problems are smooth enough.

Second order estimates for finite volume semi-discrete approximations with respect to the L^2-norm were shown in [17] and [18] provided that the solution u of the IBVP (1)–(3) satisfies the following: $u(t) \in H^3$ and $\int_0^t (\|u(s)\|_3 + \|\frac{du}{dt}(s)\|_3)\, ds < \infty$, $t \in [0,T]$. In [40], under weaker assumptions, the same convergence orders were established for a finite volume semi-discrete approximation. The authors assume that $\|u(t)\|_2$, $\int_0^t (\|u(s)\|_2^2 + s^2 \|\frac{du}{dt}(s)\|_2^2)\, ds$, $t \in [0,T]$, are finite.

Integro-differential equations (1) can be rewritten as equivalent linear differential systems: a partial differential equation involving only a time derivative and an integro-differential equation presenting only partial derivatives with respect to the space variables. This approach was used, for instance, in [19] and recently in [41] where mixed finite element methods were studied. Systems of differential equations that are equivalent to nonlinear versions of Eq. (1) for the particular case defined by (4) with a nonlinear kernel K, $K(s,t,u) = e^{-\int_s^t \gamma(u(\xi))\, d\xi}$, were considered in [7,38]. In the first work, Galerkin finite-element method with Crank–Nicolson method for time integration was analyzed, while in [38] discontinuous Galerkin finite element methods were studied.

Recently, finite difference methods (FDM) for IBVP's defined by (1) presenting the same qualitative behavior of the corresponding continuous models were proposed in [1,8,9,20,21]. Applications of integro-differential models in drug release were considered in [4,5,21].

In the present paper we study a fully discrete scheme constructed using the so-called MOL approach: the spatial discretization is defined by a standard FDM and the time integration is defined by an implicit–explicit method. The standard FDM is based on a sequence of nonuniform grids $\overline{\Omega}_H$, $H \in \Lambda$, with maximal mesh-size H_{max} converging to zero, without any restriction on the nonuniformity. It is shown that the error of the semi-discrete approximation and its gradient are second order convergent. However the truncation error induced by the spatial discretization is only of first order. The stability and convergence of the fully discrete scheme are also established.

We introduce a new convergence analysis that is different from the one introduced in [47] which is usually followed in the literature, as for instance in [40], where a finite volume approximation for the IBVP (1)–(3) was studied. The method is based on a quasi-uniform family of triangulations and the authors proved that the semi-discretization error is second order convergent with respect to the L^2-norm. This was done introducing a Ritz–Galerkin projection and splitting the semi-discretization error into the sum of two errors that are then studied separately. The same approach was followed in [11,33,44] to study the accuracy of semi-discrete finite element approximations for the solutions of the same class of integro-differential IBVP's. Second convergence order for the semi-discretization error with respect to H^1-norm was established in [6] for the one-dimensional version of (1) but following again the approach introduced by Wheeler [47].

In this paper we prove error estimates for the semi-discrete and fully discrete finite difference approximations for the solution of (1)–(3) and for its gradient. Considering a convenient representation of the semi-discretization error we avoid the split of this error and we reduce the smoothness requirements for the solution which are usually needed when such splitting approach is used. We show that, when the domain Ω is a rectangle, the error and its gradient have second convergence order while the truncation error is only of first order. This convergence order is lower when the domain presents an oblique side. Second order estimates with respect to H^1-norm are reported in the literature. For instance in [11] these estimates were obtained for finite element solutions based on piecewise quadratic elements instead of piecewise linear elements. It should be pointed out that the results, introduced in [22] for elliptic problems with smooth solutions and in [23] for problems with solutions with lower smoothness, have a central role in the proof of the main results of the present paper.

As in [23], our FDM can be seen as a lumped mass method. In fact it can be obtained considering the piecewise linear finite element on a triangulation $\mathcal{T}_H$ generated by the rectangular grid $\overline{\Omega}_H$ and applying convenient quadrature rules to each term of the variational form of the variational problem. This means that our finite difference solution can be seen as a piecewise linear finite element solution where the triangulation $\mathcal{T}_H$ does not satisfy any smoothness requirement, and so our results can be seen as supercloseness results [45]. For FDM for elliptic equations and for parabolic equations, this property is usually called supraconvergence [3,15,16,22–24,26,28,30,31,35].

The paper is organized as follows. In Section 2 we introduce the variational formulation of our problem. In Section 3 we define a semi-discrete approximation of (1)–(3) and its stability and convergence are studied. A fully discrete scheme is presented in Section 4 and its stability and convergence are analyzed. Some numerical experiments illustrating the results of this paper are presented in Section 5. Finally in Section 6 we draw some conclusions.

2. The variational problem

This section begins with the introduction of the functional spaces needed in this work and then introduces the Galerkin formulation of our IBVP. Let $\mathcal{D}$ be a bounded open set of $\mathbb{R}^2$. For $m \in \mathbb{N}_0$ we denote by $C^m(\overline{\mathcal{D}})$ the space of functions v such that $\frac{\partial^{|\alpha|} v}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$ is continuous in $\overline{\mathcal{D}}$ for $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \in \mathbb{N}_0$, $i = 1, 2$, $|\alpha| = \alpha_1 + \alpha_2 \leqslant m$. In this space we consider the following norm

$$\|v\|_{C^m(\overline{\mathcal{D}})} = \max_{|\alpha| \leqslant m} \max_{(x,y) \in \overline{\mathcal{D}}} \left| \frac{\partial^{|\alpha|} v}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(x, y) \right|.$$

For $p \in [2, +\infty[$, $W^{m,p}(\mathcal{D})$ denotes the usual Sobolev space with the semi-norm and norm given respectively by

$$|v|_{m,p} = \left(\sum_{|\alpha| = m} \left\| \frac{\partial^m v}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right\|_{0,p}^p \right)^{1/p}, \qquad \|v\|_{m,p} = \left(\sum_{|\alpha| \leqslant m} \left\| \frac{\partial^{|\alpha|} v}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right\|_{0,p}^p \right)^{1/p},$$

where

$$\left\| \frac{\partial^{|\alpha|} v}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right\|_{0,p}^p = \int_{\mathcal{D}} \left| \frac{\partial^{|\alpha|} v}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right|^p dx\, dy.$$

For $p = \infty$, we consider the norm

$$\|v\|_{m,\infty} \max_{|\alpha| \leqslant m} \operatorname*{ess\,sup}_{\mathcal{D}} \left| \frac{\partial^{|\alpha|} v}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right|.$$

By $H^m(\mathcal{D})$ we represent the Sobolev space $W^{m,2}(\mathcal{D})$ and $H^0(\mathcal{D}) = L^2(\mathcal{D})$. The norm $\|.\|_{m,2}$ is represented by $\|.\|_m$ and in $L^2(\mathcal{D})$ we consider the usual inner product $(.,.)_0$. The subspace of $H^m(\mathcal{D})$ of functions null on the boundary is denoted by $H_0^m(\mathcal{D})$.

Let V be a Banach space with respect to the norm $\|.\|_V$. We denote by $L^p(0, T; V)$, with $p \in [2, +\infty[$, the space of functions $v : (0, T) \to V$ such that

$$\|v\|_{L^p(0,T;V)} = \left(\int_0^T \|v(t)\|_V^p \, dt \right)^{1/p} \tag{5}$$

is finite. We also consider, for $m, r \in \mathbb{N}_0$, the space $W^{r,p}(0, T; V)$ of functions $v : (0, T) \to V$ such that $\frac{d^j v}{dt^j} \in L^p(0, T; V)$ for $j = 0, \dots, r$, and

$$\|v\|_{W^{r,p}(0,T;V)} := \left(\sum_{j=0}^r \int_0^T \left\| \frac{d^j v}{dt^j}(t) \right\|_V^p dt \right)^{1/p}, \tag{6}$$

is finite. When $p = 2$ this space is represented by $H^r(0, T; V)$ with $H^0(0, T; V) = L^2(0, T; V)$.

Let V be a Hilbert space with respect to the inner product $(.,.)_V$. We consider in $H^r(0,T;V)$ the inner product

$$(v,w)_{H^r(0,T;V)} := \sum_{j=0}^{r}\int_0^T \left(\frac{d^jv}{dt^j}(t), \frac{d^jw}{dt^j}(t)\right)_V dt. \tag{7}$$

By $L^\infty(0,T;V)$ we represent the space of functions $v:(0,T)\to V$ such that

$$\|v\|_{L^\infty(0,T;V)} := \operatorname*{ess\,sup}_{[0,T]} \big\|v(t)\big\|_V < \infty. \tag{8}$$

The space of functions $v:(0,T)\to V$ such that $\frac{d^jv}{dt^j}\in L^\infty(0,T;V)$ for $j=0,\dots,r$, and

$$\|v\|_{W^{r,\infty}(0,T;V)} := \max_{j=0,\dots,r} \operatorname*{ess\,sup}_{[0,T]} \left\|\frac{d^jv}{dt^j}(t)\right\|_V < \infty \tag{9}$$

is denoted by $W^{r,\infty}(0,T;V)$.

Let $L^2(0,T;H^{-1}(\Omega))$ be the dual space of $L^2(0,T;H^1(\Omega))$ where $H^{-1}(\Omega)$ denotes the dual space of $H^1(\Omega)$. We define

$$\mathcal{W}(0,T) = \left\{g\in L^2\big(0,T;H^1(\Omega)\big) \text{ such that } \frac{dg}{dt}\in L^2\big(0,T;H^{-1}(\Omega)\big)\right\},$$

which is a Hilbert space (see Theorem 25.4 of [46]).

For $f\in L^2(0,T;H^{-1}(\Omega))$ and $u_0\in L^2(\Omega)$, we consider the following variational formulation of problem (1)–(3): find $u\in\mathcal{W}(0,T)$ such that $u(t)=\psi(t)$ on $\partial\Omega$ and

$$\begin{cases} \left\langle \frac{du}{dt}(t), v\right\rangle + a\big(u(t),v\big) = \int_0^t b\big(s,t,u(s),v\big)\,ds + \big(f(t),v\big)_0 \quad \text{a.e. in } (0,T) \text{ for all } v\in H_0^1(\Omega), \\ u(0)=u_0, \end{cases} \tag{10}$$

where $\langle .,.\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, $a(.,.), b(s,t,.,.)$ are the sesquilinear forms defined respectively by

$$a(v,w) = (\mathcal{A}\nabla v,\nabla w)_0 - (\mathcal{A}_0 v,\nabla w)_0 + (a_0 v, w)_0, \quad \text{for } v,w\in H^1(\Omega), \tag{11}$$

and

$$b(s,t,v,w) = \big(\mathcal{B}(s,t)\nabla v,\nabla w\big)_0 - \big(\mathcal{B}_0(s,t)v,\nabla w\big)_0 + \big(b_0(s,t)v,w\big)_0, \quad \text{for } v,w\in H^1(\Omega). \tag{12}$$

In (11) and (12) we use the notation: $((p_1,p_2),(q_1,q_2))_0 = (p_1,q_1)_0 + (p_2,q_2)_0$, $p_i,q_i\in L^2(\Omega)$, $i=1,2$.

The coefficient functions of the integro-differential equation (1) are assumed to be smooth enough with respect to the space variables x and y, e.g. they are in $W^{m,\infty}(\Omega)$, $m\in\{1,2\}$.

3. A semi-discrete Galerkin approximation

3.1. The semi-discrete problem

In what follows we introduce the semi-discretization of (10) (see [23]). The spacial grid $\overline{\Omega}_H$ is defined by $\mathbb{R}_H\cap\overline{\Omega}$ where $H=(\mathbf{h},\mathbf{k})$, $\mathbf{h}=(h_j)_\mathbb{Z}$, $\mathbf{k}=(k_\ell)_\mathbb{Z}$ are two sequences of mesh-sizes and $\mathbb{R}_H=\mathbb{R}_\mathbf{h}\times\mathbb{R}_\mathbf{k}$ is a non-equidistant grid introduced in $\mathbb{R}^2$ with

$$\mathbb{R}_\mathbf{h} = \{x_j\in\mathbb{R}\colon x_{j+1}=x_j+h_{j+1},\ j\in\mathbb{Z}\},$$

where $x_0\in\mathbb{R}$ is given and $\mathbb{R}_\mathbf{k}$ is defined analogously with the mesh-size vector $\mathbf{k}$ in place of $\mathbf{h}$ and y_0 in place of x_0. We also introduce

$$\Omega_H := \Omega\cap\mathbb{R}_H, \qquad \partial\Omega_H := \partial\Omega\cap\mathbb{R}_H.$$

Since we are considering polygonal domains, the following compatibility condition between the grid $\overline{\Omega}_H$ and the domain Ω is assumed:

(Geom) The intersection of $\partial\Omega$ with the rectangles $\Box := (x_j,x_{j+1})\times(y_\ell,y_{\ell+1})$ spanned by points (x_j,y_ℓ), $(x_{j+1},y_{\ell+1})$ of $\mathbb{R}_H$ is either empty or it is a diagonal of $\Box$.

We consider a sequence of grids $\mathbb{R}_H$ such that the maximal mesh-size $H_{\max} := \max\{h_j, k_\ell,\ \ j, \ell \in \mathbb{Z}\}$ tends to zero. We use the symbol "Λ" for the sequence of mesh-size vectors and write "$(H \in \Lambda)$" for the convergence with respect to H running through this sequence.

By W_H we denote the space of grid functions on $\overline{\Omega}_H$ and by $W_{H,0}$ the subspace of W_H of grid functions vanishing on $\partial\Omega_H$. For convenience we assume that functions in W_H are also defined outside of $\overline{\Omega}_H$ with value equal to zero. For $(x_j, y_\ell) \in \overline{\Omega}_H$, we represent by $\square_{j,\ell}$ the box $(x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega$ where $x_{j-1/2} = x_j - \frac{h_j}{2}, x_{j+1/2} = x_j + \frac{h_{j+1}}{2}$ being $y_{\ell\pm1/2}$ defined analogously, and we denote its measure by $\omega_{j,\ell}$. Then

$$(v_H, w_H)_H := \sum_{(x_j, y_\ell) \in \overline{\Omega}_H} \omega_{j,\ell} v_{j,\ell} \overline{w}_{j,\ell}, \quad \text{for } v_H, w_H \in W_H, \tag{13}$$

defines an inner product on W_H.

By R_H we denote the operator of pointwise restriction to the grid $\overline{\Omega}_H$. Let $\mathcal{T}_H$ be a triangulation of Ω using the set $\overline{\Omega}_H$ as vertices. By $P_H v_H$ we denote the continuous piecewise linear interpolation of v_H with respect to $\mathcal{T}_H$.

The discrete version of $L^2(0, T; H^1(\Omega))$, denoted by $L^2(0, T; W_H)$, is the space of functions $w_H : [0, T] \to W_H$ such that

$$\int_0^T \|w_H(t)\|_1^2 \, dt \tag{14}$$

is finite, where $\|w_H\|_1^2 = \|w_H\|_H^2 + |P_H w_H|_1^2$ being $\|.\|_H$ the norm induced by the inner product (13) and $|.|_1$ the usual semi-norm in $H^1(\Omega)$.

Let W_H^* be the dual space of W_H and

$$\mathcal{W}_H(0, T) = \left\{ g \in L^2(0, T; W_H) \text{ such that } \frac{dg}{dt} \in L^2\big(0, T; W_H^*\big) \right\}.$$

The semi-discrete version of (10) has the form: find $u_H \in \mathcal{W}_H(0, T)$ such that $u_H(t) = R_H \psi(t)$ on $\partial\Omega_H$ and

$$\begin{cases} \left\langle \dfrac{du_H}{dt}(t), v_H \right\rangle_H + a_H\big(u_H(t), v_H\big) = \displaystyle\int_0^t b_H\big(s, t, u_H(s), v_H\big)\, ds + \big(f_H(t), v_H\big)_H \\ \quad \text{a.e. in } (0, T), \text{ for all } v_H \in W_{H,0}, \\ u_H(0) = u_{0,H}, \end{cases} \tag{15}$$

where $\langle .,. \rangle_H$ denotes the duality pairing between W_H and W_H^*, and $u_{0,H} \in W_H$ is an approximation of u_0. In (15) $a_H(\cdot,\cdot)$ and $b_H(s, t, .,.)$ are sesquilinear forms that we define in what follows.

We consider

$$a_H(.,.) = \sum_{i=1}^{2} a_{ii,H}(.,.) + \sum_{i=0}^{2} a_{i,H}(.,.) + a_{m,H}(.,.), \tag{16}$$

where $a_{ii,H}(.,.), a_{i,H}(.,.)$ are sesquilinear forms corresponding to different terms in the continuous sesquilinear form $a(.,.)$ and $a_{m,H}(.,.)$ corresponds to the mixed terms ($a_{12} = a_{21} = a_m$). The sesquilinear form $a_{11,H}(.,.)$ is defined by

$$a_{11,H}(v_H, w_H) := \sum_{\Delta \in \mathcal{T}_H} a_{11}(\Delta_x) \int_\Delta (P_H v_H)_x (P_H \overline{w}_H)_x \, dx\, dy, \tag{17}$$

where Δ_x is the midpoint of the side of $\Delta \in \mathcal{T}_H$ parallel to the x-axis. Similarly, we define $a_{22,H}(.,.)$ by

$$a_{22,H}(v_H, w_H) := \sum_{\Delta \in \mathcal{T}_H} a_{22}(\Delta_y) \int_\Delta (P_H v_H)_y (P_H \overline{w}_H)_y \, dx\, dy, \tag{18}$$

where Δ_y represents the midpoint of the side of Δ parallel to the y-axis.

The approximation of the first order terms is achieved by

$$a_{1,H}(v_H, w_H) := -\sum_{\Delta \in \mathcal{T}_H} \big[P_H(a_1 v_H)\big](\Delta_x) \int_\Delta (P_H \overline{w}_H)_x \, dx\, dy, \tag{19}$$

$$a_{2,H}(v_H, w_H) := -\sum_{\Delta \in \mathcal{T}_H} \big[P_H(a_2 v_H)\big](\Delta_y) \int_\Delta (P_H \overline{w}_H)_y \, dx\, dy. \tag{20}$$

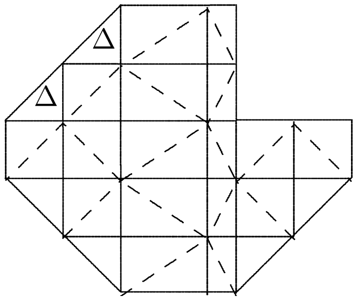

Fig. 1. Triangulation $\mathcal{T}_H^{(\nu)}$. Δ indicates triangles of $\mathcal{T}_{H,2}^{(\nu)}$.

Finally, we set

$$a_{0,H}(v_H, w_H) := \big((R_H a_0)v_H, w_H\big)_H. \tag{21}$$

The function f in the right-hand side of (1) is discretized by the grid function

$$f_H(x_j, y_\ell, t) := \frac{1}{\omega_{j,\ell}} \int_{\square_{j,\ell}} f(x, y, t)\,dx\,dy, \quad (x_j, y_\ell) \in \overline{\Omega}_H. \tag{22}$$

To define the sesquilinear form associated with the mixed derivatives, we consider two special triangulations of Ω that we call $\mathcal{T}_H^{(1)}$ and $\mathcal{T}_H^{(2)}$. They are obtained from the disjoint decomposition

$$\mathbb{R}_H = \mathbb{R}_H^{(1)} \,\dot{\cup}\, \mathbb{R}_H^{(2)},$$

where the sum $j+\ell$ of the indices of the points (x_j, y_ℓ) in $\mathbb{R}_H^{(1)}$ and in $\mathbb{R}_H^{(2)}$ is even and odd, respectively. In order to simplify the following definitions we introduce $\mathbb{R}_H^{(3)} := \mathbb{R}_H^{(1)}$. To each point $(x_j, y_\ell) \in \mathbb{R}_H$ we associate the four (open) triangles $\Delta_{j,\ell}^{(i)}$, $i = 1, 2, 3, 4$, that have an angle $\pi/2$ at (x_j, y_ℓ) and two of the four horizontal/vertical neighbor grid points of (x_j, y_ℓ) as further vertices. We then define for $\nu \in \{1, 2\}$ the triangulations

$$\mathcal{T}_{H,1}^{(\nu)} := \big\{\Delta_{j,\ell}^{(i)} \subset \Omega\colon (x_j, y_\ell) \in \mathbb{R}_H^{(\nu)},\ i \in \{1, 2, 3, 4\}\big\},$$

$$\mathcal{T}_{H,2}^{(\nu)} := \Big\{\Delta_{j,\ell}^{(i)} \subset \Big(\Omega \backslash \bigcup\big\{\Delta \,|\, \Delta \in \mathcal{T}_{H,1}^{(\nu)}\big\}\Big)\colon (x_j, y_\ell) \in \mathbb{R}_H^{(\nu+1)},\ i \in \{1, 2, 3, 4\}\Big\},$$

$$\mathcal{T}_H^{(\nu)} := \mathcal{T}_{H,1}^{(\nu)} \cup \mathcal{T}_{H,2}^{(\nu)}. \tag{23}$$

By $\mathcal{T}_H^{obl}$ we denote the set of triangles which have one side on the oblique part of $\partial\Omega$. $\mathcal{T}_H^{obl}$ is empty for a domain Ω that is union of rectangles. Fig. 1 shows an example of a triangulation $\mathcal{T}_H^{(\nu)}$ in a polygonal domain.

For $\nu = 1, 2$, the continuous piecewise linear interpolation $P_H^{(\nu)} v_H$ of a grid function $v_H \in W_H$ with respect to the triangulations $\mathcal{T}_H^{(\nu)}$ is well defined.

For each triangle $\Delta \in \mathcal{T}_H^{(\nu)}$, (x_Δ, y_Δ) denotes the vertex of Δ associated with its angle $\pi/2$, $(\tilde{x}_\Delta, y_\Delta)$ denotes the vertex that has the y-coordinate of (x_Δ, y_Δ) and $(x_\Delta, \tilde{y}_\Delta)$ denotes the other vertex of Δ. Then, for $\nu \in \{1, 2\}$, we define

$$a_m(\Delta_x) := \begin{cases} a_m(x_\Delta, y_\Delta) & \text{if } \Delta \in \mathcal{T}_{H,1}^{(\nu)}, \\ a_m(\tilde{x}_\Delta, y_\Delta) & \text{if } \Delta \in \mathcal{T}_{H,2}^{(\nu)}, \end{cases} \qquad a_m(\Delta_y) := \begin{cases} a_m(x_\Delta, y_\Delta) & \text{if } \Delta \in \mathcal{T}_{H,1}^{(\nu)}, \\ a_m(x_\Delta, \tilde{y}_\Delta) & \text{if } \Delta \in \mathcal{T}_{H,2}^{(\nu)}, \end{cases}$$

and

$$a_{m,H}(v_H, w_H) := \frac{1}{2}\big(a_{m,H}^{(1)}(v_H, w_H) + a_{m,H}^{(2)}(v_H, w_H)\big) \quad \text{for } v_H \in W_H,\ w_H \in W_{H,0}, \tag{24}$$

where

$$a_{m,H}^{(\nu)}(v_H, w_H) := \sum_{\Delta \in \mathcal{T}_H^{(\nu)}} \int_\Delta \big[a_m(\Delta_x)\big(P_H^{(\nu)} v_H\big)_x\big(P_H^{(\nu)} \overline{w}_H\big)_y + a_m(\Delta_y)\big(P_H^{(\nu)} v_H\big)_y\big(P_H^{(\nu)} \overline{w}_H\big)_x\big]\,dx\,dy.$$

The definition of the sesquilinear form

$$b_H(s,t,.,.) = \sum_{i=1}^{2} b_{ii,H}(s,t,.,.) + \sum_{i=0}^{2} b_{i,H}(s,t,.,.) + b_{m,H}(s,t,.,.) \tag{25}$$

is analogous to the definition of $a_H(.,.)$ with the convenient replacements.

The semi-discrete approximation defined by the semi-discrete variational problem (15) is obtained solving an ordinary differential system. To define such system we introduce the following finite difference operators

$$\begin{aligned} A_H v_H = {} & -\delta_x^{(1/2)}\big(a_{11}\delta_x^{(1/2)} v_H\big) - \delta_x(a_{12}\delta_y v_H) - \delta_y(a_{21}\delta_x v_H) - \delta_y^{(1/2)}\big(a_{22}\delta_y^{(1/2)} v_H\big) \\ & + \delta_x(a_1 v_H) + \delta_y(a_2 v_H) + a_0 v_H, \end{aligned} \tag{26}$$

where

$$\delta_x^{(1/2)} v_H(x_i, y_j) = \frac{v_H(x_{i+1/2}, y_j) - v_H(x_{i-1/2}, y_j)}{h_{i+1/2}},$$

$$\delta_x^{(1/2)} v_H(x_{i+1/2}, y_j) = \frac{v_H(x_{i+1}, y_j) - v_H(x_i, y_j)}{h_{i+1}},$$

$$\delta_x v_H(x_i, y_j) = \frac{v_H(x_{i+1}, y_j) - v_H(x_{i-1}, y_j)}{h_{i+1} + h_i},$$

with $h_{i+1/2} = \frac{h_i + h_{i+1}}{2}$. The corresponding operators in y-direction are defined analogously.

The finite difference operator $B_H(s,t)$ is defined as A_H with the coefficient of A replaced by the correspondent coefficients of $B(s,t)$.

If the operator A (or $B(s,t)$) contains mixed derivatives then A_H (or $B_H(s,t)$) acts, next to oblique parts of the boundary, on grid points outside $\overline{\Omega}_H$. As in [23], the missing quantities to build $A_H u_H$ (or $B_H(s,t)u_H$) are determined by auxiliary variables which are obtained by a kind of antisymmetric extension. For example, if $(x_j, y_\ell) \in \Omega_H$ is a grid point such that $(x_{j-1}, y_{\ell+1}) \notin \overline{\Omega}_H$, then the auxiliary value $u_{j-1,\ell+1}$ in the approximation of $(a_m u_x)_y$ is determined using

$$u_{j-1,\ell+1} - \psi_{j-1,\ell} = -(u_{j,\ell} - \psi_{j,\ell+1}). \tag{27}$$

Considering the procedure adopted in [3,6,23], it can be shown that the solution $u_H \in \mathcal{W}_H(0,T)$ of (15) solves the finite difference problem

$$\begin{cases} \dfrac{du_H}{dt}(t) + A_H u_H(t) = \displaystyle\int_0^t B_H(s,t) u_H(s)\,ds + f_H(t) & \text{in } \Omega_H, \\ u_H(t) = R_H \psi(t) & \text{on } \partial\Omega_H, \\ u_H(0) = u_{0,H}. \end{cases} \tag{28}$$

We assume in what follows that $a_H(.,.)$ is continuous, that is, there exists a positive constant a_c such that

$$\big|a_H(v_H, w_H)\big| \leqslant a_c \|P_H v_H\|_1 \|P_H w_H\|_1, \quad \text{for all } v_H, w_H \in W_{H,0}, \tag{29}$$

and $a_H(.,.)$ is coercive, that is, there exists a positive constant a_e and $\lambda \in \mathbb{R}$ such that

$$a_H(v_H, v_H) \geqslant a_e \|P_H v_H\|_1^2 - \lambda \|v_H\|_H^2, \quad \text{for all } v_H \in W_{H,0}. \tag{30}$$

We also suppose that $b_H(s,t,.,.)$ is bounded uniformly with respect to s,t, that is, there exists a positive constant b_c such that

$$\big|b_H(s,t,v_H,w_H)\big| \leqslant b_c \|P_H v_H\|_1 \|P_H w_H\|_1, \quad \text{for all } v_H, w_H \in W_{H,0},\ s,t \in [0,T]. \tag{31}$$

3.2. Stability analysis

In the stability analysis we consider homogeneous boundary conditions ($\psi = 0$) and we require some smoothness on the solution of the variational problem (15), namely, we assume that u_H is in $C^1([0,T]; W_{H,0})$, that is, $u_H : [0,T] \to W_{H,0}$ such that $\frac{du_H}{dt} : [0,T] \to W_{H,0}$ is continuous when we consider the norm $\|.\|_H$ in $W_{H,0}$.

The stability results, Theorems 1 and 2, are the two-dimensional versions of Theorems 1 and 2 of [6], consequently we will present only the main steps of their proofs. The upper bounds established in the next two stability results and in the results concerning the stability and convergence of the fully discrete approximation, depend on an exponential that can be unbounded in time. This means that these results hold only in bounded time intervals. For particular classes of integro-differential problems, stability upper bounds with respect to the L^2-norm that hold for long times were established, for instance, in [2,43].

Theorem 1. *Let us suppose that $a_H(.,.)$ and $b_H(s,t,.,.)$ satisfy (30) and (31), respectively. If the solution u_H of (15) is in $C^1([0,T];W_{H,0})$, then*

$$\|u_H(t)\|_H^2 + \int_0^t \|P_H u_H(s)\|_1^2\,ds \leqslant \frac{1}{\min\{1, 2(a_e-\epsilon^2)\}} e^{Ct}\left(\|u_H(0)\|_H^2 + \frac{1}{2\eta^2}\int_0^t \|f_H(s)\|_H^2\,ds\right), \tag{32}$$

for $t \in [0,T]$, where

$$C = \frac{\max\{2(\lambda+\eta^2), \frac{b_c^2 T}{2\epsilon^2}\}}{\min\{1, 2(a_e-\epsilon^2)\}}, \tag{33}$$

$\eta \neq 0$ is an arbitrary constant and $\epsilon \neq 0$ is such that

$$a_e - \epsilon^2 > 0. \tag{34}$$

Proof. From (15) with $v_H = u_H(t)$ and considering the assumptions (30) and (31) we deduce

$$\frac{1}{2}\frac{d}{dt}\|u_H(t)\|_H^2 + (a_e-\epsilon^2)\|P_H u_H(t)\|_1^2 \leqslant \frac{b_c^2}{4\epsilon^2}\left(\int_0^t \|P_H u_H(s)\|_1\,ds\right)^2 + \frac{1}{4\eta^2}\|f_H(t)\|_H^2 + (\eta^2+\lambda)\|u_H(t)\|_H^2, \tag{35}$$

where ϵ and η are non-zero constants.

From (35) we obtain

$$\frac{d}{dt}\|u_H(t)\|_H^2 + 2(a_e-\epsilon^2)\|P_H u_H(t)\|_1^2 \leqslant \frac{b_c^2 T}{2\epsilon^2}\int_0^t \|P_H u_H(s)\|_1^2\,ds + \frac{1}{2\eta^2}\|f_H(t)\|_H^2 + 2(\eta^2+\lambda)\|u_H(t)\|_H^2, \tag{36}$$

which allow us to get

$$\|u_H(t)\|_H^2 + \int_0^t \|P_H u_H(s)\|_1^2\,ds \leqslant C\int_0^t\left(\int_0^s \|P_H u_H(\mu)\|_1^2\,d\mu + \|u_H(s)\|_H^2\right)ds + \frac{1}{\min\{1, 2(a_e-\epsilon^2)\}}\left(\|u_H(0)\|_H^2 + \frac{1}{2\eta^2}\int_0^t \|f_H(s)\|_H^2\,ds\right), \tag{37}$$

with C defined by (33) and for ϵ satisfying (34). Finally applying Gronwall's lemma to (37) we obtain (32). □

Theorem 2. *Let us suppose that $a_H(.,.)$ satisfies (30) with $\lambda = 0$, $b_H(s,t,.,.)$ satisfies (31),*

$$\exists b_e > 0 \quad \text{such that} \quad b_H(t,t,v_H,v_H) \geqslant b_e\|P_H v_H\|_1^2, \tag{38}$$

for all $v_H \in W_{H,0}$, $t \in [0,T]$, and

$$\exists b_d > 0 \quad \text{such that} \quad \left|\frac{\partial b_H}{\partial t}(s,t,v_H,w_H)\right| \leqslant b_d\|P_H v_H\|_1\|P_H w_H\|_1, \tag{39}$$

for all $v_H, w_H \in W_{H,0}$, $s,t \in [0,T]$.

If the solution u_H of (15) is in $C^1([0,T];W_{H,0})$, then

$$\int_0^t \left\|\frac{du_H}{ds}(s)\right\|_H^2 ds + \|P_H u_H(t)\|_1^2 + \int_0^t \|P_H u_H(s)\|_1^2\,ds \leqslant \frac{1}{\min\{1, a_e-\eta^2, 2(b_e-\epsilon^2)\}} e^{Ct}\left(a_c\|P_H u_H(0)\|_1^2 + \int_0^t \|f_H(s)\|_H^2\,ds\right), \quad t \in [0,T], \tag{40}$$

where ϵ, η *are such that*

$$a_e - \eta^2 > 0, \qquad b_e - \epsilon^2 > 0, \tag{41}$$

and

$$C = \frac{\max\{\frac{b_c^2 T}{\eta^2}, \frac{b_d^2 T}{\epsilon^2}\}}{\min\{1, a_e - \eta^2, 2(b_e - \epsilon^2)\}}. \tag{42}$$

Proof. Equality (15) with $v_H = \frac{du_H}{dt}(t)$ can be rewritten in the following equivalent form

$$\begin{aligned}\left\|\frac{du_H}{dt}(t)\right\|_H^2 + \frac{1}{2}\frac{d}{dt}a_H\big(u_H(t), u_H(t)\big) = {}& \frac{d}{dt}\int_0^t b_H\big(s, t, u_H(s), u_H(t)\big)\,ds \\ & - \int_0^t \frac{\partial b_H}{\partial t}\big(s, t, u_H(s), u_H(t)\big)\,ds - b_H\big(t, t, u_H(t), u_H(t)\big) \\ & + \left(f_H(t), \frac{du_H}{dt}(t)\right)_H.\end{aligned} \tag{43}$$

Considering in (43) the assumptions (38), (39), it can be shown that

$$\begin{aligned}&\frac{1}{2}\left\|\frac{du_H}{dt}(t)\right\|_H^2 + \big(b_e - \epsilon^2\big)\|P_H u_H(t)\|_1^2 + \frac{1}{2}\frac{d}{dt}a_H\big(u_H(t), u_H(t)\big) \\ &\quad\leqslant \frac{d}{dt}\int_0^t b_H\big(s, t, u_H(s), u_H(t)\big)\,ds + \frac{b_d^2 T}{4\epsilon^2}\int_0^t \|P_H u_H(s)\|_1^2\,ds + \frac{1}{2}\|f_H(t)\|_H^2\end{aligned} \tag{44}$$

holds for any $\epsilon \neq 0$.

Inequality (44) leads to

$$\begin{aligned}&\int_0^t \left\|\frac{du_H}{ds}(s)\right\|_H^2 ds + 2\big(b_e - \epsilon^2\big)\int_0^t \|P_H u_H(s)\|_1^2\,ds + a_H\big(u_H(t), u_H(t)\big) \\ &\quad\leqslant 2\int_0^t b_H\big(s, t, u_H(s), u_H(t)\big)\,ds + \frac{b_d^2 T}{2\epsilon^2}\int_0^t\left(\int_0^s \|P_H u_H(\mu)\|_1^2\,d\mu\right)ds \\ &\qquad + \int_0^t \|f_H(s)\|_H^2\,ds + a_H\big(u_H(0), u_H(0)\big).\end{aligned} \tag{45}$$

Using now, in (45), the assumptions (30) with $\lambda = 0$ and (31), we obtain

$$\begin{aligned}&\int_0^t \left\|\frac{du_H}{ds}(s)\right\|_H^2 ds + 2\big(b_e - \epsilon^2\big)\int_0^t \|P_H u_H(s)\|_1^2\,ds + \big(a_e - \eta^2\big)\|P_H u_H(t)\|_1^2 \\ &\quad\leqslant \int_0^t\left(\frac{b_d^2 T}{2\epsilon^2}\int_0^s \|P_H u_H(\mu)\|_1^2\,d\mu + \frac{b_c^2 T}{\eta^2}\|P_H u_H(s)\|_1^2\right)ds \\ &\qquad + \int_0^t \|f_H(s)\|_H^2\,ds + a_c\|P_H u_H(0)\|_1^2,\end{aligned}$$

where $\eta \neq 0$ is an arbitrary constant. Consequently, for η and ϵ satisfying (41) and with C defined by (42), we establish

$$\int_0^t \left\| \frac{du_H}{ds}(s) \right\|_H^2 ds + \int_0^t \| P_H u_H(s) \|_1^2 ds + \| P_H u_H(t) \|_1^2$$
$$\leqslant C \int_0^t \left(\int_0^s \| P_H u_H(\mu) \|_1^2 d\mu + \| P_H u_H(s) \|_1^2 \right) ds$$
$$+ \frac{1}{\min\{1, a_e - \eta^2, 2(b_e - \epsilon^2)\}} \left(\int_0^t \| f_H(s) \|_H^2 ds + a_c \| P_H u_H(0) \|_1^2 \right).$$

An application of Grownwall's lemma leads to (40). □

3.3. Convergence analysis

3.3.1. The classical approach

Let $e_H(t) = R_H u(t) - u_H(t)$ be the error induced by the introduced spatial discretization. We establish in what follows a supraconvergent–superconvergent upper bound for $e_H(t)$ using the approach introduced in [47] and largely followed in the literature. In order to simplify the presentation we assume that Ω is a rectangular domain and that C is a positive constant, not depending on u and H, and that is not necessarily the same in all expressions.

Following [6,47], an estimate for $e_H(t)$ is obtained estimating $\rho_H(t) = R_H u(t) - \tilde{u}_H(t)$ and $\theta_H(t) = \tilde{u}_H(t) - u_H(t)$ with $\tilde{u}_H(t)$ defined by

$$a_H(\tilde{u}_H(t), w_H) = (g_H(t), w_H)_H, \quad w_H \in W_{H,0},$$

where

$$g_H(t) = \int_0^t (B(s,t)u(s))_H ds + f_H(t) - \left(\frac{du}{dt}(t) \right)_H,$$

being $(B(s,t)u(s))_H$ and $(\frac{du}{dt}(t))_H$ defined by (22) with f replaced by $B(s,t)u(s)$ and $\frac{du}{dt}(t)$ respectively.

An estimate for $\rho_H(t)$, depending on certain norm of $u(t)$, can be obtained considering the convergence analysis for finite difference scheme in the stationary case as for instance in [23]. In this particular case, assuming that $a_H(.,.)$ is elliptic which means that (30) holds with $\lambda = 0$, we have, for $\mu \in \{1, 2\}$,

$$\| \rho_H(t) \|_H^2 \leqslant C \| P_H \rho_H(t) \|_1^2 \leqslant C H_{max}^{2\mu} \left(\| u(t) \|_{\mu+1}^2 + \int_0^t \| u(s) \|_{\mu+1}^2 ds \right), \tag{46}$$

provided that $u \in L^\infty(0, T; H^{\mu+1}(\Omega))$, $\frac{du}{dt} \in L^\infty(0, T; L^2(\Omega))$.

It can be shown that $\theta_H(t)$ satisfies the equality

$$\left(\frac{d\theta_H}{dt}(t), v_H \right)_H + a_H(\theta_H(t), v_H) = \int_0^t b_H(s, t, e_H(s), v_H) ds - \left(\frac{d\rho_H}{dt}(t), v_H \right)_H$$
$$+ \int_0^t \left(\left((B(s,t)u(s))_H, v_H \right)_H - b_H(s, t, R_H u(s), v_H) \right) ds$$
$$+ \left(R_H \left(\frac{du}{dt}(t) \right) - \left(\frac{du}{dt}(t) \right)_H, v_H \right)_H, \quad \text{for all } v_H \in W_{H,0}. \tag{47}$$

In order to obtain an estimate for $\|\theta_H(t)\|_H^2 + \int_0^t \|P_H \theta_H(s)\|_1^2 ds$ we introduce the following notations:

$$\tau_d(v_H) = \left(R_H \frac{du}{dt}(t), v_H \right)_H - \left(\left(\frac{du}{dt}(t) \right)_H, v_H \right)_H, \tag{48}$$

and

$$\tau_{int}(v_H) = \int_0^t \left(\left((B(s,t)u(s))_H, v_H \right)_H - b_H(s, t, R_H u(s), v_H) \right) ds, \tag{49}$$

for $v_H \in W_{H,0}$.

Estimates for $\tau_d(v_H) + \tau_{int}(v_H)$ are obtained using the results presented in [23] for elliptic operators. Considering Lemmas 5.1, 5.2, 5.4, 5.5 and 5.7 of [23] we state the following proposition.

Proposition 1. *Let Ω be a rectangular domain and $\mu \in \{1, 2\}$. If the coefficients of B are in $W^{\mu,\infty}(\Omega)$ for $t, s \in [0, T]$, then, for $v_H \in W_{H,0}$, $\tau(v_H) = \tau_d(v_H) + \tau_{int}(v_H)$ satisfies*

$$\left|\tau(v_H)\right| \leqslant \tau^{(\mu)}\big(u(t)\big)\|P_H v_H\|_1,$$

where

$$\tau^{(1)}\big(u(t)\big) \leqslant C\left(\left(\sum_{\Delta\in\mathcal{T}_H} (\operatorname{diam}\Delta)^4 \left\|\frac{du}{dt}(t)\right\|^2_{H^2(\Delta)}\right)^{1/2} + \int_0^t \left(\sum_{\Delta\in\mathcal{T}_H} (\operatorname{diam}\Delta)^2 \left\|u(s)\right\|^2_{H^2(\Delta)}\right)^{1/2} ds\right)$$

$$\leqslant C H_{max}\left(\left\|\frac{du}{dt}(t)\right\|_2 + \int_0^t \left\|u(s)\right\|_2 ds\right), \tag{50}$$

provided that $u \in W^{1,\infty}(0, T; H^2(\Omega))$, and

$$\tau^{(2)}\big(u(t)\big) \leqslant C\left(\left(\sum_{\Delta\in\mathcal{T}_H} (\operatorname{diam}\Delta)^4 \left\|\frac{du}{dt}(t)\right\|^2_{H^2(\Delta)}\right)^{1/2} + \int_0^t \left(\sum_{\Delta\in\mathcal{T}_H} (\operatorname{diam}\Delta)^4 \left\|u(s)\right\|^2_{H^3(\Delta)}\right)^{1/2} ds\right)$$

$$\leqslant C H^2_{max}\left(\left\|\frac{du}{dt}(t)\right\|_2 + \int_0^t \left\|u(s)\right\|_3 ds\right), \tag{51}$$

provided that $u \in L^\infty(0, T; H^3(\Omega)) \cap W^{1,\infty}(0, T; H^2(\Omega))$. $\square$

An estimate for $\|\theta_H(t)\|_H^2 + \int_0^t \|P_H\theta_H(s)\|_1^2\, ds$ is obtained following the proof of Theorem 1. As $\theta_H(t)$ satisfies (47), it can be shown that

$$\left\|\theta_H(t)\right\|_H^2 + \int_0^t \left\|P_H\theta_H(s)\right\|_1^2 ds$$

$$\leqslant C\left(\int_0^t\int_0^s \left\|P_H e_H(\xi)\right\|_1^2 d\xi\, ds + \left\|\theta_H(0)\right\|_H^2 + \int_0^t \left(\left\|\frac{d\rho_H}{ds}(s)\right\|_H^2 + \tau^{(\mu)}\big(u(s)\big)^2\right) ds\right), \tag{52}$$

where $\tau^{(\mu)}(u(s))$ is defined in Proposition 1.

As

$$\left\|e_H(t)\right\|_H^2 + \int_0^t \left\|P_H e_H(s)\right\|_1^2 ds \leqslant 2\left(\left\|\theta_H(t)\right\|_H^2 + \int_0^t \left\|P_H\theta_H(s)\right\|_1^2 ds + \left\|\rho_H(t)\right\|_H^2 + \int_0^t \left\|P_H\rho_H(s)\right\|_1^2 ds\right),$$

we obtain, using (52),

$$\left\|e_H(t)\right\|_H^2 + \int_0^t \left\|P_H e_H(s)\right\|_1^2 ds \leqslant C\left(\int_0^t\int_0^s \left\|P_H e_H(\xi)\right\|_1^2 d\xi\, ds + \left\|\theta_H(0)\right\|_H^2 + \int_0^t \left(\left\|\frac{d\rho_H}{ds}(s)\right\|_H^2 + \tau^{(\mu)}\big(u(s)\big)^2\right) ds\right.$$
$$\left. + \left\|\rho_H(t)\right\|_H^2 + \int_0^t \left\|P_H\rho_H(s)\right\|_1^2 ds\right). \tag{53}$$

Applying Gronwall's lemma in (53) we establish

$$\left\|e_H(t)\right\|_H^2 + \int_0^t \left\|P_H e_H(s)\right\|_1^2 ds \leqslant e^{Ct} C\left(\left\|\theta_H(0)\right\|_H^2 + \int_0^t \left(\left\|\frac{d\rho_H}{ds}(s)\right\|_H^2 + \tau^{(\mu)}\big(u(s)\big)^2\right) ds\right.$$
$$\left. + \left\|\rho_H(t)\right\|_H^2 + \int_0^t \left\|P_H\rho_H(s)\right\|_1^2 ds\right). \tag{54}$$

The term $\tau^{(\mu)}(u(t))$ was estimated in Proposition 1. As for $\|\rho_H(t)\|_H^2$, for $\|\frac{d\rho_H}{dt}(t)\|_H^2$ holds the following

$$\left\|\frac{d\rho_H}{dt}(t)\right\|_H^2 \leqslant C\left\|P_H\frac{d\rho_H}{dt}(t)\right\|_0^2 \leqslant CH_{max}^{2\mu}\left(\left\|\frac{du}{dt}(t)\right\|_{\mu+1}^2 + \int_0^t \left\|\frac{du}{dt}(s)\right\|_{\mu+1}^2 ds\right), \tag{55}$$

provided that $\frac{du}{dt} \in L^\infty(0,T;H^{\mu+1}(\Omega))$ and $\frac{d^2u}{dt^2} \in L^\infty(0,T;L^2(\Omega))$.

Considering in (54) the estimates (46), (50), (51) and (55), we obtain the following estimate for $e_H(t)$

$$\|e_H(t)\|_H^2 + \int_0^t \|P_He_H(s)\|_1^2\,ds \leqslant CH_{max}^{\mu+1}, \quad \mu \in \{1,2\},$$

provided that $u \in W^{1,\infty}(0,T;H^{\mu+1}(\Omega))$, $\frac{d^2u}{dt^2} \in L^\infty(0,T;L^2(\Omega))$ and $\|\theta_H(0)\|_H^2 \leqslant CH_{max}^{\mu+1}$ and C depending on u.

3.3.2. A new approach

We introduce in what follows a new approach that permit us to reduce the smoothness required for $u(t)$ with respect to that used before. We start by noting that $e_H(t)$ satisfies the equality

$$\frac{1}{2}\frac{d}{dt}\|e_H(t)\|_H^2 = \left(R_H\frac{du}{dt}(t), e_H(t)\right)_H + a_H\big(u_H(t), e_H(t)\big) - \int_0^t b_H\big(s,t,u_H(s),e_H(t)\big)\,ds - \big(f_H(t), e_H(t)\big)_H. \tag{56}$$

As

$$\big(f_H(t), e_H(t)\big)_H = \left(\left(\frac{du}{dt}(t)\right)_H, e_H(t)\right)_H + \left(\left(Au(t) - \int_0^t B(s,t)u(s)\,ds\right)_H, e_H(t)\right)_H, \tag{57}$$

where $(\frac{du}{dt}(t))_H$, $(Au(t) - \int_0^t B(s,t)u(s)\,ds)_H$ are defined by (22) with $f(t)$ replaced by $\frac{du}{dt}(t)$ and $Au(t) - \int_0^t B(s,t)u(s)\,ds$, respectively, from (56) we obtain

$$\frac{1}{2}\frac{d}{dt}\|e_H(t)\|_H^2 + a_H\big(e_H(t), e_H(t)\big) = \int_0^t b_H\big(s,t,e_H(s),e_H(t)\big)\,ds + \tau\big(e_H(t)\big), \tag{58}$$

where

$$\tau\big(e_H(t)\big) = \tau_d\big(e_H(t)\big) + \tau_A\big(e_H(t)\big) + \tau_{int}\big(e_H(t)\big), \tag{59}$$

with $\tau_d(e_H(t))$ and $\tau_{int}(e_H(t))$ defined by (48) and (49), respectively, with v_H replaced by $e_H(t)$, and

$$\tau_A\big(e_H(t)\big) = a_H\big(R_Hu(t), e_H(t)\big) - \big((Au(t))_H, e_H(t)\big)_H. \tag{60}$$

An estimate for $\tau_d(e_H(t)) + \tau_{int}(e_H(t))$, when Ω is a rectangle, is obtained from Proposition 1. The following Proposition 2 leads to an estimate for $\tau(e_H(t))$ defined by (59). As Proposition 1, Proposition 2 is established considering Lemmas 5.1, 5.2, 5.4, 5.5 and 5.7 of [23]. Let $\tau(v_H)$ be defined by (59) with $e_H(t)$ replaced by $v_H \in W_{H,0}$. By Ω_H^{obl} we denote the following set $\Omega_H^{obl} = \bigcup\{\Delta \mid \Delta \in \mathcal{T}_H^{obl}\}$.

Proposition 2. *Let the grids $\overline{\Omega}_H$, with $H \in \Lambda$, satisfy the condition* (Geom) *and consider $\mu \in \{1,2\}$. If the coefficients of A and $B(s,t)$ are in $W^{\mu,\infty}(\Omega)$ for $t,s \in [0,T]$, then, for $v_H \in W_{H,0}$, $\tau(v_H)$ satisfies*

$$\big|\tau(v_H)\big| \leqslant \tau^{(\mu)}\big(u(t)\big)\|P_Hv_H\|_1,$$

where

$$\begin{aligned}\tau^{(1)}\big(u(t)\big) &\leqslant C\left(\left(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^2\|u(t)\|_{H^2(\Delta)}^2\right)^{1/2} + \left(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\left\|\frac{du}{dt}(t)\right\|_{H^2(\Delta)}^2\right)^{1/2}\right.\\ &\qquad\left. + \int_0^t\left(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^2\|u(s)\|_{H^2(\Delta)}^2\right)^{1/2} ds\right)\\ &\leqslant CH_{max}\left(\|u(t)\|_2 + \left\|\frac{du}{dt}(t)\right\|_2 + \int_0^t\|u(s)\|_2\,ds\right),\end{aligned} \tag{61}$$

provided that $u \in W^{1,\infty}(0,T;H^2(\Omega))$, *and*

$$\tau^{(2)}\big(u(t)\big) \leqslant C\left(\left(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\big\|u(t)\big\|^2_{H^3(\Delta)}\right)^{1/2}+\left(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\left\|\frac{du}{dt}(t)\right\|^2_{H^2(\Delta)}\right)^{1/2}\right.$$
$$\left.+\int_0^t\left(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\big\|u(s)\big\|^2_{H^3(\Delta)}\right)^{1/2}ds\right)$$
$$+\sigma_{mix}\left(\left(\sum_{\Delta\in\mathcal{T}_H^{obl}}(\operatorname{diam}\Delta)^{4(1-1/p)}\big|u(t)\big|^2_{W^{2,p}(\Delta)}\right)^{1/2}\right.$$
$$\left.+\int_0^t\left(\sum_{\Delta\in\mathcal{T}_H^{obl}}(\operatorname{diam}\Delta)^{4(1-1/p)}\big|u(s)\big|^2_{W^{2,p}(\Delta)}\right)^{1/2}ds\right)$$
$$\leqslant CH^2_{max}\left(\big\|u(t)\big\|_3+\left\|\frac{du}{dt}(t)\right\|_2+\int_0^t\big\|u(s)\big\|_3\,ds\right)$$
$$+C\sigma_{mix}H^{3/2-1/p}_{max}\left(\big|u(t)\big|_{W^{2,p}(\Omega_H^{obl})}+\int_0^t\big|u(s)\big|_{W^{2,p}(\Omega_H^{obl})}\,ds\right), \tag{62}$$

provided that $u \in L^\infty(0,T;H^3(\Omega))\cap W^{1,\infty}(0,T;H^2(\Omega))$ *and* $p\in[2,\infty)$.

If Ω *has an oblique side and* $a_m \neq 0$ *or* $b_m \neq 0$, *then, in* (62), $\sigma_{mix}=1$. *Otherwise, if* Ω *is a rectangle or* $a_m=b_m=0$, *then, in* (62), $\sigma_{mix}=0$. □

We state now one of the main results of this paper.

Theorem 3. *Let the grids* $\overline{\Omega}_H$, *with* $H\in\Lambda$, *satisfy the condition* (Geom) *and consider* $\mu\in\{1,2\}$. *If the coefficients of* A *and* $B(s,t)$ *are in* $W^{\mu,\infty}(\Omega)$ *for* $t,s\in[0,T]$, *and* $a_H(.,.)$ *and* $b_H(s,t,.,.)$ *satisfy respectively* (30) *and* (31), *then*

$$\big\|e_H(t)\big\|^2_H+\int_0^t\big\|P_He_H(s)\big\|^2_1\,ds$$
$$\leqslant\frac{1}{\min\{1,2(a_e-\epsilon^2-\eta^2)\}}e^{\tilde{C}t}\left(\big\|e_H(0)\big\|^2_H+\frac{1}{2\eta^2}\int_0^t g^{(\mu)}(\varsigma)^2\,d\varsigma\right), \tag{63}$$

where ϵ *and* η *are non-zero constants such that*

$$a_e-\epsilon^2-\eta^2>0, \tag{64}$$

and

$$\tilde{C}=\frac{\max\{2\lambda,\frac{Tb_c^2}{2\epsilon^2}\}}{\min\{1,2(a_e-\epsilon^2-\eta^2)\}}, \tag{65}$$

$$g^{(1)}(t)^2=C\left(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^2\left(\|u\|^2_{L^\infty(0,T;H^2(\Delta))}+\left\|\frac{du}{dt}\right\|^2_{L^\infty(0,T;H^2(\Delta))}+\|u\|^2_{L^2(0,t;H^2(\Delta))}\right)\right)$$
$$\leqslant CH^2_{max}\left(\|u\|^2_{L^\infty(0,T;H^2(\Omega))}+\left\|\frac{du}{dt}\right\|^2_{L^\infty(0,T;H^2(\Omega))}+\|u\|^2_{L^2(0,T;H^2(\Omega))}\right), \tag{66}$$

provided that $u\in W^{1,\infty}(0,T;H^2(\Omega))$,

$$g^{(2)}(t)^2 = C\Bigg(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\Bigg(\|u\|^2_{L^\infty(0,T;H^3(\Delta))} + \left\|\frac{du}{dt}\right\|^2_{L^\infty(0,T;H^2(\Delta))} + \|u\|^2_{L^2(0,t;H^3(\Delta))}\Bigg)$$
$$+ \sigma_{mix}\sum_{\Delta\in\mathcal{T}_H^{obl}}(\operatorname{diam}\Delta)^{4(1-1/p)}\big(\|u\|^2_{L^\infty(0,T;W^{2,p}(\Delta))} + \|u\|^2_{L^2(0,t;W^{2,p}(\Delta))}\big)\Bigg)$$
$$\leqslant CH^4_{max}\Bigg(\|u\|^2_{L^\infty(0,T;H^3(\Omega))} + \left\|\frac{du}{dt}\right\|^2_{L^\infty(0,T;H^2(\Omega))} + \|u\|^2_{L^2(0,t;H^3(\Omega))}\Bigg)$$
$$+ C\sigma_{mix}H^{3-2/p}_{max}\big(\|u\|^2_{L^\infty(0,T;W^{2,p}(\Omega_H^{obl}))} + \|u\|^2_{L^2(0,t;W^{2,p}(\Omega_H^{obl}))}\big), \tag{67}$$

provided that $u \in L^\infty(0,T;H^3(\Omega)) \cap W^{1,\infty}(0,T;H^2(\Omega))$ and $p \in [2,\infty)$.

If Ω has an oblique side and $a_m \neq 0$ or $b_m \neq 0$, then, in (67), $\sigma_{mix} = 1$. Otherwise, if Ω is a rectangle or $a_m = b_m = 0$, then, in (67), $\sigma_{mix} = 0$.

Proof. Considering in (58) the assumptions (30) and (31) for $a_H(.,.)$ and $b_H(s,t,.,.)$, respectively, we deduce

$$\frac{d}{dt}\|e_H(t)\|^2_H + 2\big(a_e - \epsilon^2 - \eta^2\big)\|P_He_H(t)\|^2_1 \leqslant \frac{Tb_c^2}{2\epsilon^2}\int_0^t \|P_He_H(s)\|^2_1\,ds + 2\lambda\|e_H(t)\|^2_H + \frac{1}{2\eta^2}\tau^{(\mu)}\big(u(t)\big)^2,$$

and consequently

$$\|e_H(t)\|^2_H + \int_0^t\|P_He_H(s)\|^2_1\,ds \leqslant \tilde{C}\int_0^t\Bigg(\int_0^s\|P_He_H(\upsilon)\|^2_1\,d\upsilon + \|e_H(s)\|^2_H\Bigg)ds$$
$$+ \frac{1}{\min\{1, 2(a_e-\epsilon^2-\eta^2)\}}\Bigg(\frac{1}{2\eta^2}\int_0^t g^{(\mu)}(s)^2\,ds + \|e_H(0)\|^2_H\Bigg), \tag{68}$$

for ϵ and η satisfying (64) and with $\tilde{C}$ defined by (65). Applying Gronwall's lemma to inequality (68) we conclude (63). □

Remark 1. Considering Corollary 6.2 of [23], under the assumptions of Theorem 3, if $u \in L^\infty(0,T;C^2(\overline{\Omega}\cup\Omega_0))$, where Ω_0 is a neighborhood of the oblique part of $\partial\Omega$, we can state the following estimate for $g^{(2)}(t)$

$$g^{(2)}(t)^2 \leqslant C\Bigg(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\Bigg(\|u(t)\|^2_{H^3(\Delta)} + \left\|\frac{du}{dt}(t)\right\|^2_{H^2(\Delta)} + \int_0^t\|u(s)\|^2_{H^3(\Delta)}\,ds\Bigg)$$
$$+ \sigma_{mix}\sum_{\Delta\in\mathcal{T}_H^{obl}}(\operatorname{diam}\Delta)^4\Bigg(\|u(t)\|^2_{C^2(\overline{\Delta})} + \int_0^t\|u(s)\|^2_{C^2(\overline{\Delta})}\,ds\Bigg)\Bigg)$$
$$\leqslant CH^4_{max}\Bigg(\|u(t)\|^2_3 + \left\|\frac{du}{dt}(t)\right\|^2_2 + \int_0^t\|u(s)\|^2_3\,ds\Bigg)$$
$$+ C\sigma_{mix}H^3_{max}\Bigg(\|u(t)\|^2_{C^2(\Omega_H^{obl})} + \int_0^t\|u(s)\|^2_{C^2(\Omega_H^{obl})}\,ds\Bigg)$$
$$\leqslant CH^4_{max}\Bigg(\|u\|^2_{L^\infty(0,T;H^3(\Omega))} + \left\|\frac{du}{dt}\right\|^2_{L^\infty(0,T;H^2(\Omega)} + \|u\|^2_{L^2(0,t;H^3(\Omega))}\Bigg)$$
$$+ C\sigma_{mix}H^3_{max}\big(\|u\|^2_{L^\infty(0,T;C^2(\Omega_H^{obl}))} + \|u\|^2_{L^2(0,T;C^2(\Omega_H^{obl}))}\big), \tag{69}$$

where it was assume that $\sum_{\Delta\in\mathcal{T}_H^{obl}}\operatorname{diam}\Delta \leqslant C$.

4. A fully discrete approximation

4.1. The fully discrete variational problem

We introduce in $[0, T]$ a uniform grid $\{t_n,\ n = 0, \ldots, N\}$ with $t_0 = 0$, $t_N = T$ and $t_n - t_{n-1} = \Delta t$. By D_{-t} we denote the backward finite difference operator with respect to time variable. Let u_H^n be the fully discrete approximation in W_H such that $u_H^n = R_H \psi(t_n)$ on $\partial\Omega_H$ and

$$\begin{cases} \left(D_{-t}u_H^{n+1}, v_H\right)_H + a_H\left(u_H^{n+1}, v_H\right) = \Delta t \sum_{\ell=0}^{n} b_H\left(t_\ell, t_{n+1}, u_H^\ell, v_H\right) + \left(f_H^{n+1}, v_H\right)_H, \\ \quad n = 0, \ldots, N-1,\ \forall v_H \in W_{H,0}, \\ u_H^0 = u_{0,H}. \end{cases} \tag{70}$$

We remark that $u_H^n \in W_H$ satisfying (70) is also a solution of the fully discrete finite difference problem

$$\begin{cases} D_{-t}u_H^{n+1} + A_H u_H^{n+1} = \Delta t \sum_{\ell=0}^{n} B_H(t_\ell, t_{n+1}) u_H^\ell + f_H^{n+1} & \text{in } \Omega_H,\ n = 0, \ldots, N-1, \\ u_H^n = R_H \psi(t_n) & \text{on } \partial\Omega_H,\ n = 1, \ldots, N, \\ u_H^0 = u_{0,H}, \end{cases} \tag{71}$$

which defines an implicit–explicit scheme to solve numerically (1), (2), (3). In fact (71) can be established combining the spatial discretization introduced in the previous sections with the left rectangular rule to discretize the time integral.

In certain cases, the method (71) can be rewritten as a three-time-level method. In fact, for $n \geqslant 1$, we have

$$D_{-t}u_H^{n+1} + A_H u_H^{n+1} - f_H^{n+1} = \Delta t B_H(t_n, t_{n+1}) u_H^n + \Delta t \sum_{\ell=0}^{n-1} B_H(t_\ell, t_{n+1}) u_H^\ell$$

and

$$D_{-t}u_H^n + A_H u_H^n - f_H^n = \Delta t \sum_{\ell=0}^{n-1} B_H(t_\ell, t_n) u_H^\ell.$$

Moreover if

$$B_H(t_\ell, t_{n+1}) u_H^\ell = g(\Delta t) B_H(t_\ell, t_n) u_H^\ell, \tag{72}$$

then

$$D_{-t}u_H^{n+1} + A_H u_H^{n+1} - f_H^{n+1} = \Delta t B_H(t_n, t_{n+1}) u_H^n + g(\Delta t)\left(D_{-t}u_H^n + A_H u_H^n - f_H^n\right),$$

which has the form of a three-time-level method. This approach allow a drastic reduction of the computational cost when compared with method (71). Note that the condition (72) is satisfied, for instance, when $B(s, t)u(t) = K(t - s)Bu(t)$ and $K(a + b) = K(a)K(b)$.

4.2. Stability and convergence analysis

We study in what follows the qualitative behavior of the solution of (71) (or (70)). An essential tool is the following lemma.

Lemma 1. *(Discrete Gronwall inequality (Lemma 4.3 of [11]).) Let $\{\eta_n\}$ be a sequence of nonnegative real numbers satisfying*

$$\eta_n \leqslant \sum_{j=0}^{n-1} \omega_j \eta_j + \beta_n \quad \text{for } n \geqslant 1,$$

where $\omega_j \geqslant 0$ and $\{\beta_n\}$ is a nondecreasing sequence of nonnegative numbers. Then

$$\eta_n \leqslant \beta_n \exp\left(\sum_{j=0}^{n-1} \omega_j\right) \quad \text{for } n \geqslant 1. \tag{73}$$

Theorem 4. *Under the assumptions of Theorem 1, the solution of (70) satisfies*

$$\|u_H^n\|_H^2 + \Delta t \sum_{m=0}^{n} \|P_H u_H^m\|_1^2 \leqslant \tilde{C}\left(\|u_H^0\|_H^2 + 2(a_e - \epsilon^2)\Delta t \|P_H u_H^0\|_1^2 + \frac{\Delta t}{2\eta^2}\sum_{m=1}^{n}\|f_H^m\|_H^2\right) \tag{74}$$

where $\eta \neq 0$, $\epsilon \neq 0$, ϵ is such that

$$a_e - \epsilon^2 > 0, \tag{75}$$

the time step size Δt satisfies

$$1 - 2(\lambda + \eta^2)\Delta t > 0, \tag{76}$$

and

$$\tilde{C} = \frac{\exp\left(\frac{T \max\{2(\lambda+\eta^2), \frac{b_c^2 T}{2\epsilon^2}\}}{\min\{1-2(\eta^2+\lambda)\Delta t, 2(a_e-\epsilon^2)\}}\right)}{\min\{1 - 2(\lambda + \eta^2)\Delta t, 2(a_e - \epsilon^2)\}}.$$

Proof. Using $n = m$, $v_H = u_H^{m+1}$, in (70), the coercivity (30) of $a_H(.,.)$ and the uniform continuity (31) of $b_H(s,t,.,.)$, we establish

$$(D_{-t}u_H^{m+1}, u_H^{m+1})_H + a_e\|P_H u_H^{m+1}\|_1^2 - \lambda\|u_H^{m+1}\|_H^2 \leqslant b_c \Delta t \sum_{j=0}^{m}\|P_H u_H^j\|_1 \|P_H u_H^{m+1}\|_1 + (f_H^{m+1}, u_H^{m+1})_H. \tag{77}$$

As we have

$$b_c \Delta t \sum_{j=0}^{m}\|P_H u_H^j\|_1 \|P_H u_H^{m+1}\|_1 \leqslant \frac{b_c^2 T \Delta t}{4\epsilon^2}\sum_{j=0}^{m}\|P_H u_H^j\|_1^2 + \epsilon^2\|P_H u_H^{m+1}\|_1^2,$$

and

$$(f_H^{m+1}, u_H^{m+1})_H \leqslant \frac{1}{4\eta^2}\|f_H^{m+1}\|_H^2 + \eta^2\|u_H^{m+1}\|_H^2,$$

for all $\epsilon \neq 0$, $\eta \neq 0$, from (77) we deduce

$$\begin{aligned}
&\|u_H^{m+1}\|_H^2 - \|u_H^m\|_H^2 + 2\Delta t(a_e - \epsilon^2)\|P_H u_H^{m+1}\|_1^2 \\
&\quad \leqslant \frac{b_c^2 T \Delta t^2}{2\epsilon^2}\sum_{j=0}^{m}\|P_H u_H^j\|_1^2 + \Delta t \frac{1}{2\eta^2}\|f_H^{m+1}\|_H^2 + 2(\lambda + \eta^2)\Delta t\|u_H^{m+1}\|_H^2.
\end{aligned} \tag{78}$$

Summing (78) over $m = 0, \ldots, n-1$, we get

$$\begin{aligned}
&\|u_H^n\|_H^2 - \|u_H^0\|_H^2 + 2\Delta t(a_e - \epsilon^2)\sum_{m=0}^{n-1}\|P_H u_H^{m+1}\|_1^2 \\
&\quad \leqslant \frac{b_c^2 T \Delta t^2}{2\epsilon^2}\sum_{m=0}^{n-1}\sum_{j=0}^{m}\|P_H u_H^j\|_1^2 + \frac{\Delta t}{2\eta^2}\sum_{m=0}^{n-1}\|f_H^{m+1}\|_H^2 + 2(\lambda + \eta^2)\Delta t \sum_{m=0}^{n-1}\|u_H^{m+1}\|_H^2,
\end{aligned}$$

and consequently

$$\begin{aligned}
&(1 - 2(\lambda + \eta^2)\Delta t)\|u_H^n\|_H^2 + 2\Delta t(a_e - \epsilon^2)\sum_{m=0}^{n}\|P_H u_H^m\|_1^2 \\
&\quad \leqslant \|u_H^0\|_H^2 + 2\Delta t(a_e - \epsilon^2)\|P_H u_H^0\|_1^2 + \frac{\Delta t}{2\eta^2}\sum_{m=1}^{n}\|f_H^m\|_H^2 \\
&\qquad + \sum_{m=0}^{n-1}\frac{b_c^2 T \Delta t}{2\epsilon^2}\Delta t \sum_{j=0}^{m}\|P_H u_H^j\|_1^2 + 2(\lambda + \eta^2)\Delta t \sum_{m=1}^{n-1}\|u_H^m\|_H^2.
\end{aligned} \tag{79}$$

Choosing in (79) Δt, ϵ and η satisfying (75) and (76) we obtain

$$\|u_H^n\|_H^2+\Delta t\sum_{m=0}^{n}\|P_Hu_H^m\|_1^2 \leqslant \sum_{m=0}^{n-1}C\left(\|u_H^m\|_H^2+\Delta t\sum_{j=0}^{m}\|P_Hu_H^j\|_1^2\right) + \frac{1}{\min\{1-2(\lambda+\eta^2)\Delta t,2(a_e-\epsilon^2)\}}\left(\|u_H^0\|_H^2+2\Delta t\big(a_e-\epsilon^2\big)\|P_Hu_H^0\|_1^2+\frac{\Delta t}{2\eta^2}\sum_{m=1}^{n}\|f_H^m\|_H^2\right), \tag{80}$$

with

$$C=\frac{\Delta t\max\{2(\lambda+\eta^2),\frac{b_c^2T}{2\epsilon^2}\}}{\min\{1-2(\lambda+\eta^2)\Delta t,2(a_e-\epsilon^2)\}}.$$

Finally an application of the discrete Gronwall's lemma leads to (74). □

The stability of (71) is now established.

Theorem 5. *Under the assumptions of Theorem 1, for the solution u_H^n of (71), with $f_H^{n+1}=0$, holds the following inequality*

$$\|u_H^n\|_H^2+\Delta t\sum_{m=0}^{n}\|P_Hu_H^m\|_1^2\leqslant\tilde{C}\big(\|u_H^0\|_H^2+2\big(a_e-\epsilon^2\big)\Delta t\|P_Hu_H^0\|_1^2\big) \tag{81}$$

with

$$\tilde{C}=\frac{\exp(\frac{T\max\{2\lambda,\frac{b_c^2T^2}{2\epsilon^2}\}}{\min\{1-2\lambda\Delta t_0,2(a_e-\epsilon^2)\}})}{\min\{1-2\lambda\Delta t_0,2(a_e-\epsilon^2)\}},$$

for $\epsilon\neq0$ satisfying (75) and $\Delta t\in(0,\Delta t_0)$, where Δt_0 is such that

$$1-2\lambda\Delta t_0>0. \quad \square \tag{82}$$

Since for λ nonpositive we conclude the stability of (71) without any condition on the time step size Δt, that is the method is unconditionally stable. Otherwise, it is conditionally stable.

Let $e_H^n=R_Hu(t_n)-u_H^n$ be the error for the solution u_H^n defined by (71). An estimation for this error is established in the next result.

Theorem 6. *Under the assumptions of Theorem 1, if $\frac{\partial b_H}{\partial s}(s,t,.,.)$ is uniformly continuous*

$$\left|\frac{\partial b_H}{\partial s}(s,t,u_H,v_H)\right|\leqslant b_d\|P_Hu_H\|_1\|P_Hv_H\|_1,\quad \forall u_H,v_H\in W_{H,0},\ s,t\in[0,T], \tag{83}$$

then there exists a positive constant C which does not depend on H, Δt and u, such that the error $e_H^n=R_Hu(t_n)-u_H^n$, with u_H^n defined by (71) (or (70)), satisfies the following

$$\|e_H^n\|_H^2+\Delta t\sum_{m=0}^{n}\|P_He_H^m\|_1^2 \leqslant\tilde{C}\left(2\Delta t\big(a_e-\epsilon^2-\gamma_2^2-\gamma_3^2\big)\|P_He_H^0\|_1^2+\|e_H^0\|_H^2+\Delta t\sum_{m=1}^{n}\frac{1}{2\gamma_3^2}g^{(\mu)}(t_m)^2 + C\Delta t^2\left(\frac{1}{2\gamma_1^2}\|R_Hu\|^2_{H^2(0,T;W_H)}+\frac{b_f^2T}{2\gamma_2^2}\|P_HR_Hu\|^2_{H^1(0,T;H^1(\Omega))}\right)\right), \tag{84}$$

where

$$\tilde{C}=\frac{\exp(\frac{T\max\{2(\lambda+\gamma_1^2),\frac{b_f^2T}{2\epsilon^2}\}}{\min\{1-2\Delta t_0(\lambda+\gamma_1^2),2(a_e-\epsilon^2-\gamma_2^2-\gamma_3^2)\}})}{\min\{1-2(\lambda+\gamma_1^2)\Delta t_0,2(a_e-\epsilon^2-\gamma_2^2-\gamma_3^2)\}},$$

$b_f = \max\{b_c, b_d\}$, ϵ, $\gamma_i \neq 0$, $i = 1, 2, 3$, are such that

$$a_e - \epsilon^2 - \gamma_2^2 - \gamma_3^2 > 0,$$

and $\Delta t \in (0, \Delta t_0)$, with Δt_0 fixed by

$$1 - 2\big(\lambda + \gamma_1^2\big)\Delta t_0 > 0. \tag{85}$$

In (84), $g^{(\mu)}(t_m)$ for $\mu \in \{1, 2\}$, is defined by (66) and (67), respectively, for $\mu = 1$ and $\mu = 2$ with $t = t_m$.

Proof. It is easy to show that

$$\begin{aligned}\big(D_{-t}e_H^{m+1}, e_H^{m+1}\big)_H &= \big(D_{-t}R_H u(t_{m+1}), e_H^{m+1}\big)_H + a_H\big(u_H^{m+1}, e_H^{m+1}\big)\\ &\quad - \Delta t\sum_{j=0}^{m} b_H\big(t_j, t_{m+1}, u_H^j, e_H^{m+1}\big) - \big(f_H^{m+1}, e_H^{m+1}\big)_H.\end{aligned} \tag{86}$$

Considering that (57) holds with $t = t_{m+1}$, from (86), we deduce

$$\big(D_{-t}e_H^{m+1}, e_H^{m+1}\big)_H + a_H\big(e_H^{m+1}, e_H^{m+1}\big) = \Delta t\sum_{j=0}^{m} b_H\big(t_j, t_{m+1}, e_H^j, e_H^{m+1}\big) + \tau_{cd}\big(e_H^{m+1}\big) \tag{87}$$

with

$$\tau_{cd}\big(e_H^{m+1}\big) = \tau\big(e_H^{m+1}\big) + \tau_n\big(e_H^{m+1}\big),$$

where $\tau(e_H^{m+1})$ is defined by (59) with $e_H(t)$ replaced by e_H^{m+1},

$$\tau_n\big(e_H^{m+1}\big) = \tau_{n,1}\big(e_H^{m+1}\big) + \tau_{n,2}\big(e_H^{m+1}\big),$$

and

$$\begin{aligned}\tau_{n,1}\big(e_H^{m+1}\big) &= \left(D_{-t}R_H u(t_{m+1}) - R_H\frac{du}{dt}(t_{m+1}), e_H^{m+1}\right)_H,\\ \tau_{n,2}\big(e_H^{m+1}\big) &= \int_0^{t_{m+1}} b_H\big(s, t_{m+1}, R_H u(s), e_H^{m+1}\big)\,ds - \Delta t\sum_{j=0}^{m} b_H\big(t_j, t_{m+1}, R_H u(t_j), e_H^{m+1}\big).\end{aligned} \tag{88}$$

We remark that an estimate for $\tau(e_H^{m+1})$ is obtained considering Proposition 2. For $\tau_{n,1}(e_H^{m+1})$ we have

$$\big|\tau_{n,1}\big(e_H^{m+1}\big)\big| \leqslant C\int_{t_m}^{t_{m+1}}\left\|R_H\frac{d^2u}{dt^2}(s)\right\|_H ds\big\|e_H^{m+1}\big\|_H \leqslant C\Delta t\frac{1}{4\gamma_1^2}\|R_H u\|^2_{H^2(t_m,t_{m+1};W_H)} + \gamma_1^2\big\|e_H^{m+1}\big\|_H^2, \tag{89}$$

where $\gamma_1 \neq 0$ is an arbitrary constant.

The estimate for $\tau_{n,2}(e_H^{m+1})$

$$\big|\tau_{n,2}\big(e_H^{m+1}\big)\big| \leqslant C\Delta t\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\left(\left|\frac{\partial b_H}{\partial s}\big(s, t_{m+1}, R_H u(s), e_H^{m+1}\big)\right| + \left|b_H\left(s, t_{m+1}, R_H\frac{du}{dt}(s), e_H^{m+1}\right)\right|\right)ds, \tag{90}$$

is obtained using the Bramble–Hilbert Lemma. As $b_H(s, t, ., .)$ and $\frac{\partial b_H}{\partial s}(s, t, ., .)$ are uniformly continuous, from (90), we obtain

$$\begin{aligned}\big|\tau_{n,2}\big(e_H^{m+1}\big)\big| &\leqslant C\Delta t b_f\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\left(\big\|P_H R_H u(s)\big\|_1 + \left\|P_H R_H\frac{du}{dt}(s)\right\|_1\right)ds\big\|P_H e_H^{m+1}\big\|_1\\ &\leqslant \frac{1}{4\gamma_2^2}C\Delta t^2 b_f^2\|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))} + \gamma_2^2\big\|P_H e_H^{m+1}\big\|_1^2,\end{aligned} \tag{91}$$

where $\gamma_2 \neq 0$ is an arbitrary constant.

Combining the estimations (89), (91) with the estimates for $\tau(e_H^{m+1})$ obtained considering Proposition 2, we get

$$\tau_{cd}\big(e_H^{m+1}\big) \leqslant \frac{1}{4\gamma_3^2} g^{(\mu)}(t_{m+1})^2 + \big(\gamma_3^2+\gamma_2^2\big)\big\|P_H e_H^{m+1}\big\|_1^2 + \gamma_1^2\big\|e_H^{m+1}\big\|_H^2 + C\left(\frac{1}{4\gamma_1^2}\Delta t\|R_H u\|^2_{H^2(t_m,t_{m+1};W_H)} + \frac{1}{4\gamma_2^2} b_f^2 \Delta t^2 \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))}\right), \tag{92}$$

where $\mu \in \{1,2\}$, $g^{(1)}(t_{m+1})^2$ and $g^{(2)}(t_{m+1})^2$ are given by (66) and (67), respectively, with $t = t_{m+1}$.

From (87) and (92), it can be deduced, following the proof of Theorem 4, that the errors e_H^j, $j = 0, \ldots, m+1$, satisfy

$$\begin{aligned}
&\big\|e_H^{m+1}\big\|_H^2 - \big\|e_H^m\big\|_H^2 + 2\Delta t\big(a_e - \epsilon^2 - \gamma_2^2 - \gamma_3^2\big)\big\|P_H e_H^{m+1}\big\|_1^2 \\
&\quad\leqslant \Delta t^2 \frac{b_f^2 T}{2\epsilon^2}\sum_{j=0}^{m}\big\|P_H e_H^j\big\|_1^2 + 2\Delta t\big(\lambda+\gamma_1^2\big)\big\|e_H^{m+1}\big\|_H^2 + \Delta t\frac{1}{2\gamma_3^2} g^{(\mu)}(t_{m+1})^2 \\
&\qquad + C\Delta t\left(\frac{1}{2\gamma_1^2}\Delta t\|R_H u\|^2_{H^2(t_m,t_{m+1};W_H)} + \frac{b_f^2}{2\gamma_2^2}\Delta t^2\|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))}\right),
\end{aligned} \tag{93}$$

which leads, following again the proof of Theorem 4, to (84). □

Remark 2. Assumption (83) holds, for instance, for $B(s,t)u(t) = K(t-s)Bu(t)$ where the kernel is such that $|K'(t-s)| \leqslant C$, $t, s \in [0,T]$ and B is a second order differential operator such that $b(.,.)$ is continuous.

Corollary 1. *Under the assumptions of Theorem 1 and taking $u_{0,H} = R_H u_0$, there exists a positive constant C which does not depend on H and Δt and u, such that, for $\Delta t \in (0, \Delta t_0)$, with Δt_0 verifying (85), the error $e_H^n = R_H u(t_n) - u_H^n$, with u_H^n defined by (71), satisfies the following*

$$\big\|e_H^n\big\|_H^2 + \Delta t\sum_{m=1}^{n}\big\|P_H e_H^m\big\|_1^2 \leqslant C\big(H_{max}^2\big(\|u\|^2_{W^{1,\infty}(0,T;H^2(\Omega))} + \|u\|^2_{L^2(0,T;H^2(\Omega))}\big) + \Delta t^2\big(\|R_H u\|^2_{H^2(0,T;W_H)} + \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))}\big)\big), \tag{94}$$

provided that $u \in W^{1,\infty}(0,T;H^2(\Omega)) \cap H^2(0,T;C(\Omega))$, and

$$\begin{aligned}
\big\|e_H^n\big\|_H^2 + \Delta t\sum_{m=1}^{n}\big\|P_H e_H^m\big\|_1^2 \leqslant{}& C\big(H_{max}^4\big(\|u\|^2_{W^{1,\infty}(0,T;H^2(\Omega))} + \|u\|^2_{L^\infty(0,T;H^3(\Omega))} + \|u\|^2_{L^2(0,T;H^3(\Omega))}\big) \\
&+ \sigma_{mix} H_{max}^{3-2/p}\big(\|u\|^2_{L^\infty(0,T;W^{1,p}(\Omega_H^{obl}))} + \|u\|^2_{L^2(0,T;W^{1,p}(\Omega_H^{obl}))}\big) \\
&+ \Delta t^2\big(\|R_H u\|^2_{H^2(0,T;W_H)} + \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))}\big)\big),
\end{aligned} \tag{95}$$

provided that $u \in W^{1,\infty}(0,T;H^3(\Omega)) \cap H^2(0,T;C(\Omega))$ and $p \in [2,\infty)$.

If Ω has an oblique side and $a_m \neq 0$ or $b_m \neq 0$, then, in (95), $\sigma_{mix} = 1$. Otherwise, if Ω is a rectangle or $a_m = b_m = 0$, then, in (95), $\sigma_{mix} = 0$. □

Remark 3. Considering Corollary 6.2 of [23], under the assumptions of Corollary 1, if the coefficients functions are in $W^{2,\infty}(\Omega)$, $u \in L^\infty(0,T;C^2(\overline{\Omega}\cup\Omega_0))$, where Ω_0 is a neighborhood of the oblique part of $\partial\Omega$, then we can state the following estimate

$$\begin{aligned}
\big\|e_H^n\big\|_H^2 + \Delta t\sum_{m=1}^{n}\big\|P_H e_H^m\big\|_1^2 \leqslant{}& C\big(H_{max}^4\big(\|u\|^2_{W^{1,\infty}(0,T;H^2(\Omega))} + \|u\|^2_{L^\infty(0,T;H^3(\Omega))} + \|u\|^2_{L^2(0,T;H^3(\Omega))}\big) \\
&+ \sigma_{mix} H_{max}^3\big(\|u\|^2_{L^\infty(0,T;C^2(\Omega_H^{obl}))} + \|u\|^2_{L^2(0,T;C^2(\Omega_H^{obl}))}\big) \\
&+ \Delta t^2\big(\|R_H u\|^2_{H^2(0,T;W_H)} + \|P_H R_H u\|^2_{H^1(0,T;H^1(\Omega))}\big)\big).
\end{aligned} \tag{96}$$

Table 1
Convergence rates.

H_{max}	N_x	N_y	E_H^N	$R_{H1,H2}$
1.500×10^{-1}	10	11	9.437×10^{-4}	1.860
7.500×10^{-2}	20	22	2.600×10^{-4}	1.944
3.750×10^{-2}	40	44	6.759×10^{-5}	1.980
1.875×10^{-2}	80	88	1.713×10^{-5}	1.992
9.375×10^{-3}	160	176	4.305×10^{-6}	2.000
4.688×10^{-3}	320	352	1.076×10^{-6}	2.000
2.344×10^{-3}	640	704	2.691×10^{-7}	–

5. Numerical simulation

In this section we illustrate the theoretical results obtained for the integro-differential IBVP (1)–(3).

Example 1. Let Ω be defined by $\Omega = (0,1) \times (0,1)$. We consider the IBVP (1)–(3) with

$$\mathcal{A}(x,y) = \begin{bmatrix} 1 & xy \\ xy & 1 \end{bmatrix}, \qquad \mathcal{A}_0(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad a_0(x,y) = -1,$$

$$\mathcal{B}(s,t,x,y) = e^{-(t-s)} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathcal{B}_0(s,t,x,y) = 0, \qquad b_0(s,t,x,y) = 0.$$

The boundary conditions, the initial condition and the term f are such that this problem has the solution

$$u(x,y,t) = e^t xy(x-1)(y-1), \quad (x,y) \in \overline{\Omega},\ t \in [0, 0.1]. \tag{97}$$

The numerical solutions are obtained using the method (70) with a uniform time grid in $[0, 0.1]$ with step size $\Delta t = 10^{-6}$. In the spatial domain, we consider an initial random partition with $H_{max} = 0.15$ and $N_x = 10$, $N_y = 11$ points in x and y axis, respectively. We use grids in the consecutive computations that are defined by introducing the midpoint in each interval $[x_i, x_{i+1}]$ and $[y_j, y_{j+1}]$ of the previous grid. We measure in the simulations the error

$$E_H^M = \left(\|e_H^M\|_H^2 + \Delta t \sum_{j=1}^{M} \|P_H e_H^j\|_1^2 \right)^{1/2},$$

where the errors e_H^j, $j = 0, \ldots, M$, are computed considering the numerical solution and the known solution (97).

The error E_H^M as well as H_{max}, for each partition $\overline{\Omega}_H$, the number of points N_x and N_y, and the rate R_{H_1,H_2}

$$R_{H_1,H_2} = \frac{\ln(\frac{E^M_{H_{1,max}}}{E^M_{H_{2,max}}})}{\ln(\frac{H_{1,max}}{H_{2,max}})}$$

are presented in Table 1.

The results presented in Table 1 show that the error E_H^M is of second order in H_{max}. This fact illustrates the estimate (96).

Example 2. Let Ω be the polygonal domain presented in Fig. 2. We consider the IBVP (1)–(3), with

$$\mathcal{A}(x,y) = \begin{bmatrix} 1 & xy \\ xy & 1 \end{bmatrix}, \qquad \mathcal{A}_0(x,y) = 0, \qquad a_0(x,y) = 0,$$

$$\mathcal{B}(s,t,x,y) = e^{-(t-s)} \begin{bmatrix} 0 & -xy \\ -xy & 0 \end{bmatrix}, \qquad \mathcal{B}_0(s,t,x,y) = 0, \qquad b_0(s,t,x,y) = 0.$$

The boundary conditions, the initial condition and the term f are such that this problem has the following solution

$$u(x,y,t) = e^t xy(x-1)(y-1)\left(-x + \frac{7}{5} - y\right), \quad (x,y) \in \overline{\Omega},\ t \in [0, 0.1].$$

In the time interval $[0, 0.1]$ we consider a grid with step size $\Delta t = 10^{-6}$. We introduce in $\overline{\Omega}$ an initial nonuniform grid $\overline{\Omega}_H$ satisfying the condition (Geom). The grids used in the numerical experiments are defined using the procedure introduced in Example 1. The error E_H^M, the rate R_{H_1,H_2} as well as H_{max}, for each partition $\overline{\Omega}_H$, the number of points N_x and N_y, are presented in Table 2 (the notations used were introduced in Example 1).

The results presented in Table 2 illustrate the error estimate (96).

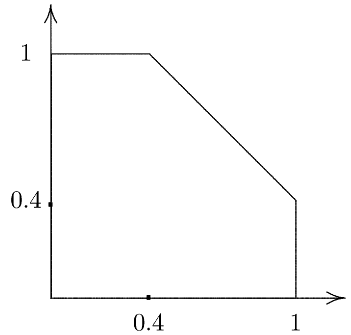

Fig. 2. Polygonal domain.

Table 2
Convergence rates.

H_{max}	N_x	N_y	E_H^M	R_{H_1,H_2}
1.347×10^{-1}	9	8	2.988×10^{-4}	1.554
6.733×10^{-2}	18	16	1.018×10^{-4}	1.560
3.367×10^{-2}	36	32	3.451×10^{-5}	1.539
1.683×10^{-2}	72	64	1.188×10^{-5}	1.522
8.416×10^{-3}	144	128	4.136×10^{-6}	1.511
4.208×10^{-3}	288	256	1.451×10^{-6}	1.506
2.104×10^{-3}	576	512	5.108×10^{-7}	–

6. Conclusions

In this paper numerical methods for the IBVP (1)–(3) were proposed. The methods were defined using MOL approach, that is, they were defined combining a spatial discretization, which converts the integro-differential problem in an ordinary differential problem, with a time integration method of the implicit–explicit type. The semi-discrete solution was studied and a supraconvergence result was established. The stability and the convergence of the fully discrete method were also studied. In the convergence analysis we introduced a different approach from the one that is usually followed in the literature (see for instance [42,44,47,48]). Such new approach enable us to assume lower smoothness of the solution of the IBVP (1)–(3), than those that we need to assume if the approach introduced in [47] was followed.

The methods studied can be seen into different class of methods: the class of Galerkin methods and the class of finite difference methods. In fact, with respect to the spatial discretization, the methods were constructed considering the variational formulation of the differential problem and replacing the space $H_0^1(\Omega)$ by the space of the piecewise linear functions and using convenient quadrature rules.

We point out that the analysis presented here can be followed if we use in the time integration methods of higher order such as Crank–Nicolson method. This remark holds if we replace the rectangular rule, considered in the approximation of the time integral, by higher approximation methods.

Acknowledgements

The authors gratefully acknowledge the support of this work by Centro de Matemática da Universidade de Coimbra and Fundação para a Ciência e a Tecnologia, through European program COMPETE/FEDER and by the project UTAustin/MAT/0066/2008.

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