Two-layered Belnapian logics for uncertainty

Marta Bílková*

The Czech Academy of Sciences, Institute of Computer Science, Prague
bilkova@cs.cas.cz

Reasoning about information, its potential incompleteness, uncertainty, and contradictoriness need to be dealt with adequately. Separately, these characteristics have been taken into account by various appropriate logical formalisms and (classical) probability theory. While incompleteness and uncertainty are typically accommodated within one formalism, e.g. within various models of imprecise probability, contradictoriness and uncertainty less so — conflict or contradictoriness of information is rather chosen to be resolved than to be reasoned with. To reason with conflicting information, positive and negative support—evidence in favour and evidence against—a statement are quantified separately in the semantics. This two-dimensionality gives rise to logics interpreted over twist-product algebras or bi-lattices, the well known Belnap-Dunn logic of First Degree Entailment being a prominent example [2, 8]. Belnap-Dunn logic with its double-valuation frame semantics can in turn be taken as a base logic for defining various uncertainty measures on de Morgan algebras, e.g. Belnapian (non-standard) probabilities [11] or belief functions [15, 6].

In a spirit similar to Belnap-Dunn logic, we can introduce many-valued logics suitable to reason about such uncertainty measures. They are interpreted over twist-product algebras based on the $[0, 1]$ real interval as their standard semantics and can be seen to account for the two-dimensionality of positive and negative component of (the degree of) belief or likelihood based on potentially contradictory information, quantified by an uncertainty measure. The logics presented in this talk include expansions of Łukasiewicz or Gödel logic with a de-Morgan negation which swaps between the positive and negative semantical component. The expansions of Gödel logic, which can be equipped with a natural double-valuation frame semantics, relate to the extensions of Nelson's paraconsistent logic $N4$ [12, 13], or Wansing's paraconsistent logic I_4C_4 [14], with the prelinearity axiom. The resulting logics inherit both (finite) standard completeness properties, and decidability and complexity properties of Łukasiewicz or Gödel logic respectively, and allow for an efficient reasoning using the constraint tableaux calculi formalism [3].

Two-layered logics for reasoning under uncertainty of classical events were introduced in [9, 10], and developed further within an abstract algebraic framework by [7] and [1]. They separate two layers of reasoning: the inner layer consists of a logic chosen to reason about events or evidence, the connecting modalities are interpreted by a chosen uncertainty measure on propositions of the inner layer, typically a probability or a belief function, and the outer layer consists of a logical framework to reason about probabilities or beliefs. The modalities apply to inner level formulas only, to produce outer level atomic formulas, and they do not nest. Logics introduced in [9] use classical propositional logic on the inner layer, and reasoning with linear inequalities on the outer layer. [10] on the other hand use Łukasiewicz logic on the outer layer, to capture the quantitative reasoning about probabilities within a propositional logical language.

Our main objective is to utilise the apparatus of two-layered modal logics for the formalisation of reasoning with uncertain information, which itself might be non-classical, i.e., incomplete or contradictory. Many-valued logics with a two-dimensional semantics mentioned above are

*This talk is rooted in joint work with S. Frittella, D. Kozhemiachenko, O. Majer and S. Nazari.

used on the outer layer to reason about belief, likelihood or certainty based on potentially incomplete or contradictory evidence, building on Belnap-Dunn logic of First Degree Entailment as an inner logic of the underlying evidence. This results in two-layered logics suitable for various scenarios: expansions of Łukasiewicz logic are adequate in cases when aggregated evidence yields a Belnapian probability measure [4] or a belief function (on a De Morgan algebra) [6], while expansions of Gödel logic are useful to reason about comparative uncertainty in cases where it is not so, or to capture reasoning about qualitative probability [5].

References

[1] P. Baldi, P. Cintula, and C. Noguera. On two-layered modal logics for uncertainty. Manuscript, 2020.

[2] N.D. Belnap. How a computer should think. In H. Omori and H. Wansing, editors, *New Essays on Belnap-Dunn Logic*, volume 418 of *Synthese Library (Studies in Epistemology, Logic, Methodology, and Philosophy of Science)*. Springer, Cham, 2019.

[3] M. Bílková, S. Frittella, and D. Kozhemiachenko. Constraint tableaux for two-dimensional fuzzy logics. In Anupam Das and Sara Negri, editors, *Automated Reasoning with Analytic Tableaux and Related Methods*, pages 20–37. Springer International Publishing, 2021.

[4] M. Bílková, S. Frittella, O. Majer, and S. Nazari. Belief based on inconsistent information. In Manuel A. Martins and Igor Sedlár, editors, *Dynamic Logic. New Trends and Applications*, pages 68–86, Cham, 2020. Springer International Publishing.

[5] M. Bílková, S. Frittella, D. Kozhemiachenko, O. Majer, and S. Nazari. Comparing certainty in contradictory evidence, 2022. Submitted.

[6] M. Bílková, S. Frittella, D. Kozhemiachenko, O. Majer, and S. Nazari. Reasoning with belief functions over Belnap–Dunn logic. https://arxiv.org/abs/2203.01060, 2022.

[7] P. Cintula and C. Noguera. Modal logics of uncertainty with two-layer syntax: A general completeness theorem. In *Proceedings of WoLLIC 2014*, pages 124–136, 2014.

[8] J. Michael Dunn. Intuitive semantics for first-degree entailments and 'coupled trees'. *Philosophical Studies*, 29(3):149–168, 1976.

[9] R. Fagin, J.Y. Halpern, and N. Megiddo. A logic for reasoning about probabilities. *Information and computation*, 87(1–2):78–128, 1990.

[10] P. Hájek, L. Godo, and F. Esteva. Fuzzy logic and probability. In Philippe Besnard and Steve Hanks, editors, *UAI '95: Proceedings of the Eleventh Annual Conference on Uncertainty in Artificial Intelligence, Montreal, Quebec, Canada, August 18-20, 1995*, pages 237–244. Morgan Kaufmann, 1995.

[11] D. Klein, O. Majer, and S. Rafiee Rad. Probabilities with gaps and gluts. *Journal of Philosophical Logic*, 2021.

[12] D. Nelson. Constructible falsity. *The Journal of Symbolic Logic*, 14(1):16–26, 1949.

[13] D. Vakarelov. Notes on N-lattices and constructive logic with strong negation. *Studia logica*, 36(1–2):109–125, 1977.

[14] H. Wansing. Constructive negation, implication, and co-implication. *Journal of Applied Non-Classical Logics*, 18(2–3):341–364, 2008.

[15] C. Zhou. Belief functions on distributive lattices. *Artificial Intelligence*, 201:1–31, 2013.