Noetherian Spaces, Wqos, and their Statures

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Well-quasi-orders (wqos) are an amazingly useful notion, both in mathematics and in computer science. In a seminal paper, de Jongh and Parikh [DeJP77] showed that maximal order type of any well-partial-order (wpo) is attained, and computed it in a few cases. In her now classic Habilitationsschrift, Schmidt [Sch79] computed it in the important cases of spaces of words, and of finite trees over wpos. Those pioneering works have had considerable influence in logic (ordinal analysis) and in computer science (verification), at least.

Fifteen years ago, the first author realized that there was a natural topological generalization of well-quasi-orders, *Noetherian spaces* [Gou07]. A space is Noetherian if and only if every open subset is compact, and the notion has many equivalent definitions. It so turns out that the special kind of Noetherian spaces whose topology is Alexandroff are, in a precise sense, exactly the wqos. Over the years, it has been observed that many results and constructions that are typical of wqo theory generalize to the Noetherian setting. The goal of this presentation is to explain some recent results of ours that extend the well-known theory of maximal order types to a corresponding theory of *statures* of Noetherian space [GLL22]. Explicitly, we define the stature of a Noetherian space X as the ordinal rank of X in the lattice $\mathcal{H}X$ of all closed subsets of X, ordered by inclusion. We argue that this notion of stature coincides with maximal order types in the case of wpos, following [Kří97] or [BG08] (from whom we borrowed the term "stature"), while a more naive idea for extending the notion of maximal order type fails. We also argue that many results on maximal order types of various wpo constructions transfer to Noetherian spaces (coproducts, products, spaces of finite words, of finite multisets), with the same formulae, and we obtain new formulae for statures of a variety of Noetherian constructions that do not arise from wpos (spaces of words with the so-called prefix topology, spaces with the cofinite topology, spaces of transfinite words [GLHL22], powersets).

Instead of spending too much time on the technical details, we focus on giving a gentle introduction to the required theory of Noetherian spaces, especially seen through the lens of a computer scientist working in verification, as in [Gou10]. With this view, our hope is that statures of Noetherian spaces would be the first step in understanding the complexity of verification of so-called topological well-structured transition systems, mimicking and extending the use of maximal order types done until now (see [FFSS11, SS11], for example). Importantly, some tools that have been developed in this theory, and most notably the theory of S-representations of [FG20], initially invented in order to produce effective completions of (standard, wqo-theoretic) well-structured transition systems and generalize the so-called Karp-Miller algorithm [KM67], are crucial here, as they provide us with concrete representations of elements of $\mathcal{H}X$, allowing us to compute appropriate lower and upper bounds on their ordinal ranks.

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