From $\{0,1\}$ to $[0,1]$: A survey of duality theorems

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The category of finite powers of the two-element set $\{0,1\}$ together with all functions between them are a Lawvere theory whose category of set-based models is the variety of Boolean algebras. The Lawvere theory is then dually equivalent to the full subcategory of free finitely generated Boolean algebras, and this duality lifts to Stone Duality between Stone spaces and Boolean algebras. Next, let us order $\{0,1\}$ as $0 < 1$, its finite powers by the product order, and let us restrict morphisms to the monotone functions. The resulting Lawvere theory is obtained from the previous one by removing some operations, notably negation. It has distributive lattices as category of set-based models, it is dually equivalent to the full subcategory of free finitely generated distributive lattices, and this duality lifts to Priestley duality between Priestley spaces and distributive lattices.

Replace $\{0,1\}$ with the real unit interval $[0,1] \subseteq \mathbb{R}$, and consider the category T' of its Cartesian powers up to some fixed, sufficiently large infinite cardinal, with all functions between them. Then T' is a Lawvere-Linton theory which provides a convenient setting to discuss a host of duality theorems—some old, some new, and some (the vast majority) uninteresting. Any subcategory T of T' that includes at least all finite powers (=finite arities) and all projection functions (=variables) provides a Lawvere-Linton theory that conceptually corresponds to a choice of structure on the Cartesian powers. Because of the inclusion $\{0,1\} \hookrightarrow [0,1]$, the dualising possibilities offered by the two-element set are subsumed; Stone and Priestley duality, for instance, may each be recovered by the appropriate choice of T. More generally, for any such T one can, in principle, study the associated category of models and its duality theory. This study, though, can be expected to hold interest only insofar as the implied structure on the Cartesian powers does, for instance in light of how it relates to mathematical tradition.

Starting from this perspective I will make an attempt to survey what is known about some cases of interest, arranging them into a hierarchy of theories. The best known case is possibly that of all continuous functions, that is, of Stone-Gelfand-Yosida duality. Two further cases of interest, each rooted in tradition to different degrees of depth, are piecewise-linear functions, yielding affine Baker-Beynon duality for compact polyhedra, and monotone continuous functions, yielding Nachbin's compact ordered spaces and a duality for them. The more recent results I plan to discuss are due to various teams of authors.