General and standard modal fuzzy logics

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Fuzzy logics are those whose algebraic semantics are classes of bounded integral residuated lattices with a continuous monoidal operation. There are three main subvarieties which allow to generate any other algebra in the class though the so-called ordinal sum construction. These are the varieties of Gödel ($\mathbb{G}$), MV ($\mathbb{MV}$) and Product ($\mathbb{P}$) algebras, algebraic semantics of Gödel (G), Łukasiewicz (Ł) and Product (Π) logics respectively. The subdirectly irreducible members of each one of these varieties are the linearly ordered algebras (chains). On the other hand, each one of these varieties is generated by a single algebra of its corresponding class, named the standard algebra, whose universe is $[0, 1]$. Moreover, the generated variety coincides with the generated quasi-variety. Summing up, each one of the previous logics is complete with respect to the 1-assertional logic of the corresponding class of chains and that of the standard algebra.

The F.O. extensions of the previous logics behave, however, differently. The logics arising from F.O. models evaluated over all algebras in the corresponding variety do coincide with those arising from F.O. models evaluated over the corresponding chains. In the literature, these are called the *general logics*. The *standard logics* are those arising from F.O. models evaluated over the corresponding standard algebra. In the Gödel case, the general logic coincides with the standard one. However, this is not the case for the Łukasiewicz nor Product logics. That F.O. general and standard Łukasiewicz logics are different follows as a corollary from the fact that the set of theorems of the general logic is recursively enumerable (R.E), but the set of theorems of the standard logic is not [4]. The same thing can be proven for the product case.

Modal fuzzy logics can be understood as the restriction of the previous F.O. logics to the fragment resulting from the usual translation of modal operators (and formulas in variables $\mathcal{V}$) to the formulas in the predicates language $\{R/2\} \cup \{P/1 \colon P \in \mathcal{V}\}$ as is done in the classical case. This approach yields the so-called valued Kripke models, which are Kripke models where the accessibility relation and the variables at each world are evaluated over an algebra like the ones above. The modal logics resulting from these semantics are the so-called *modal fuzzy logics*. It is relevant to note that, over the same class of models, two modal logics (i.e., consequence relations) are defined: the local and the global one. The latter is defined analogously to the F.O. entailment over arbitrary formulas (closing both premises and conclusions under universal quantifiers), while the former one refers to the notion of truth-entailment under each assignment into the model. Nevertheless, their sets of theorems coincide.

Analogously to the F.O. case, we can refer to the *general* or *standard modal fuzzy logics* whenever the evaluation is considered over all algebras (or equivalently, chains) of the corresponding variety, or only over the standard one. Furthermore, the particular cases when the accessibility relation in the Kripke model is taken as a classical binary relation (crisp) are also of special interest, since the underlying Kripke frames are classical.

In this talk, we will compare the previous general and standard logics. While in the Gödel case, the F.O. behavior immediately implies that, in all cases, the general and standard modal Gödel logics coincide, we will see how for the other two logics, the results are more varied. The global modal logics behave as the F.O. ones, namely, the general and standard logics differ. For the crisp-accessibility cases, a reasoning similar to the one from F.O. can be done. Indeed, we know that global modal standard Łukasiewicz and product logics with crisp accessibility relation are not R.E. [5]. However, the general F.O. Łukasiewicz and Product logics are axiomatizable. Henceforth, the corresponding general modal logics are R.E., implying that the standard and general logics do not coincide. On the other hand, the computational classification of the global logics with valued accessibility is not known. Nevertheless, two examples can be built to prove that also these logics differ, exploiting peculiarities of a model over the Chang algebra for the Łukasiewicz logic and of models over the analogous product algebra for the Product logic.

For what concerns local modal logics, however, we will see that the general and standard logics coincide, both for the crisp accessibility and for the valued one. This implies that the theorems of these logics (which are the same as the ones from the global logics) coincide too. In the Łukasiewicz cases, this equality can be proven relying in the F.O. completeness with respect to witnessed models (those in which, for each quantified formula, there is an assignment in the model where the formula without the quantifier takes the same value as the quantified one), both for arbitrary models and also for standard ones [1]. For the Product logic, we can prove the claim for the models with valued accessibility relation by relying in the details of a proof of decidability of the Description Logic over the standard product algebra [3]. This does not serve to tackle the case with crisp accessibility, which can nevertheless be proven by a different approach using the completeness of F.O. product logic with respect to models evaluated over a certain algebra (the one arising via Cignoli-Torrens functor from the lexicographic sum $\mathbb{R}^{Q}$) [2]. Using models valued over this algebra we identify certain conditions, that can be expressed with finitely many propositional formulas, and that capture all relevant information about the modal operations. In this fashion, the modal general product logic can be faithfully encoded within the the propositional one, and so, it is possible to rely in the standard completeness of the latter to prove that the general and standard modal logics also coincide.

Furthermore, this latter proof also will allow us to answer positively to the open question of the decidability of (local) crisp-accessibility modal standard product logic.

References

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