Extention theory and the calculus of butterflies

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Università degli Studi di Milano

(joint work with G. Metere)

Categorical Methods in Algebra and Topology
workshop in honour of Manuela Sobral on the occasion of her 70th birthday
Coimbra, January 26, 2014

Internal crossed modules

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$$G_0 \flat G \xrightarrow{\ \xi\ } G \xrightarrow{\ \partial\ } G_0$$

such that the following squares commute:

$$\begin{array}{ccc} G \flat G & \xrightarrow{\chi_G} & G \\ {\scriptstyle \partial\flat 1}\downarrow & & \| \\ G_0 \flat G & \xrightarrow{\ \xi\ } & G \\ {\scriptstyle 1\flat\partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ G_0 \flat G_0 & \xrightarrow[\chi_{G_0}]{} & G_0 \end{array}$$

A morphism of crossed modules $(\partial', \xi') \to (\partial, \xi)$ is a pair (f, f_0) of maps that makes the following diagram commute:

$$
\begin{array}{ccc}
H_0 \flat H & \xrightarrow{f_0 \flat f} & G_0 \flat G \\
{\scriptstyle \xi'} \downarrow & & \downarrow {\scriptstyle \xi} \\
H & \xrightarrow{f} & G \\
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$$\begin{CD}
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@V{\xi'}VV @VV{\xi}V \\
H @>{f}>> G \\
@V{\partial'}VV @VV{\partial}V \\
H_0 @>>{f_0}> G_0
\end{CD}$$

These data form a category $\mathrm{XMod}(\mathcal{C})$ equivalent to $\mathrm{Grpd}(\mathcal{C})$ [G. Janelidze '03].

This equivalence extends to a biequivalence of bicategories [Abbad, Mantovani, Metere, Vitale '13].

We can define homotopy invariants:

$$\begin{array}{ccc}
\pi_1(\partial') & \xrightarrow{\pi_1(\underline{f})} & \pi_1(\partial) \\
{\scriptstyle \ker(\partial')}\downarrow & & \downarrow{\scriptstyle \ker(\partial)} \\
H & \xrightarrow{f} & G \\
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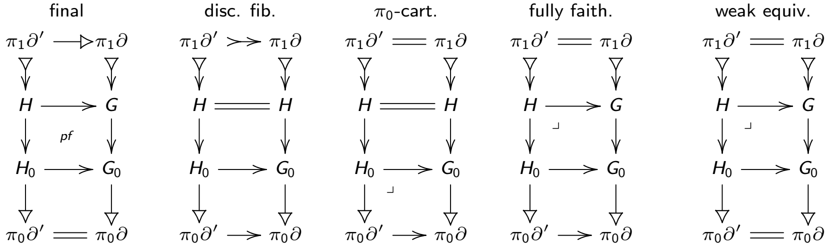
$\pi_1(\partial)$ is central in $\mathcal{C}$.
Bourn's global direction of a groupoid translates in terms of crossed modules as:

$$
\begin{array}{ccc}
\pi_1(\partial) & = & \pi_1(\partial) \\
\big\downarrow & & \big\downarrow{\scriptstyle 0} \\
G \rtimes_\xi G_0 & \longrightarrow & \pi_0(\partial)
\end{array}
$$

Translation of some special morphisms:



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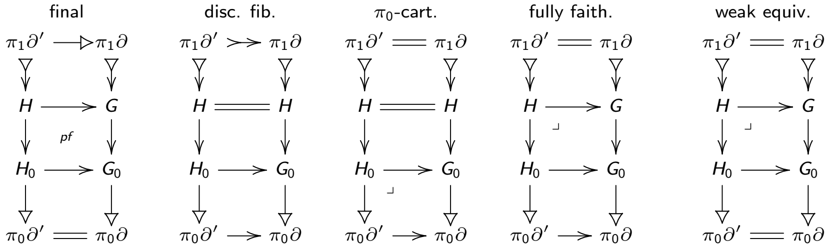

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We have (among others) two factorization systems:

$$\begin{array}{cc} (\text{final}, & \text{disc. fib.}) \\ \cap & \cup \\ (\pi_0\text{-inv.}, & \pi_0\text{-cart.}) \end{array}$$

Internal butterflies

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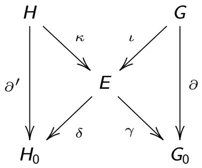
A butterfly $\widehat{E}\colon (\partial', \xi') \nrightarrow (\partial, \xi)$ is a commutative diagram of the form



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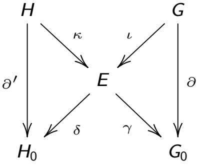

such that

i. (κ, γ) is a complex, i.e. $\gamma \cdot \kappa = 0$,
ii. (ι, δ) is short exact,
iii. The action of E on H induced by that of H_0 on H via δ makes $\kappa \colon H \to E$ a crossed module,
iv. The action of E on G induced by that of G_0 on G via γ makes $\iota \colon g \to E$ a crossed module.

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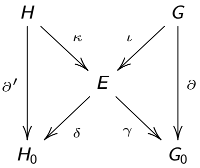

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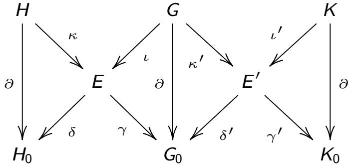
A morphism of butterflies $\widehat{E}, \widehat{E}'\colon (\partial', \xi') \nrightarrow (\partial, \xi)$ is an arrow $\alpha\colon E \to E'$ commuting with the κ's, the ι's, the δ's and the γ's.

Horizontal composition:



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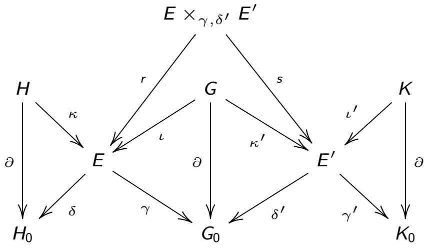

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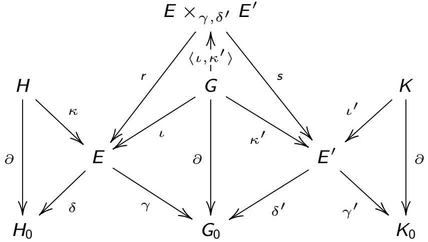

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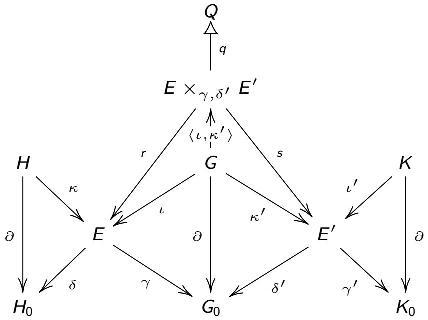

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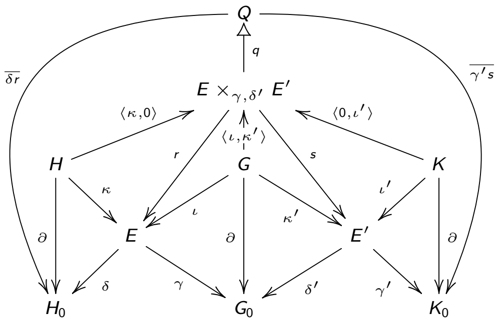

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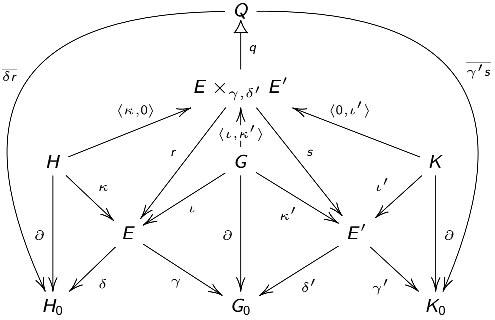
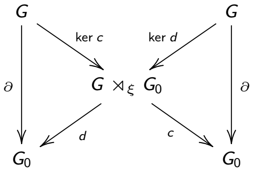


Identity butterfly:



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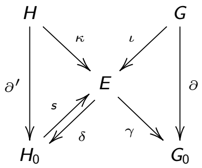

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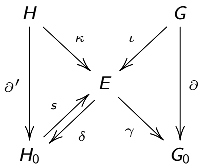
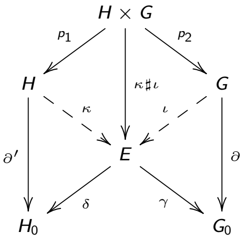


In fact, $\mathsf{Bfly}(\mathcal{C})$ is the bicategory of fractions of $\mathsf{XMod}(\mathcal{C})$ with respect to weak equivalences.

Every butterfly is associated with a span of crossed module morphisms:



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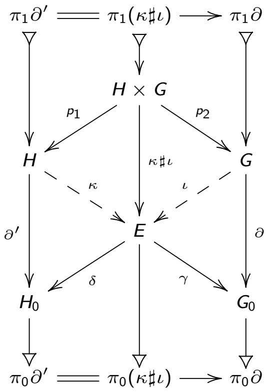

where the morphism (p_1, δ) is a weak equivalence.

This allows us to extend the definition of π_0 and π_1 to butterflies.

Extensions

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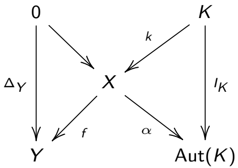

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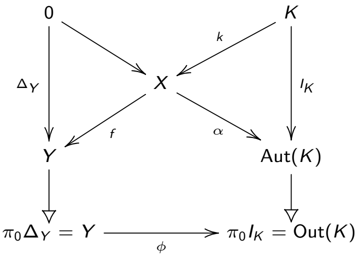

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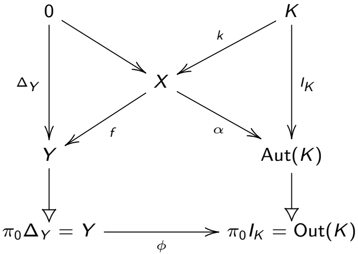

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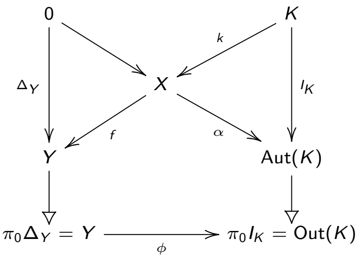

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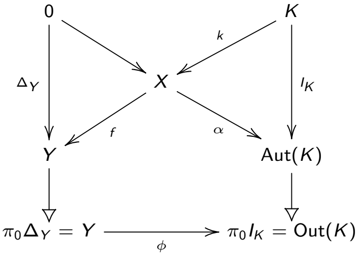

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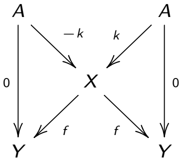
Alternative embedding for abelian extensions:



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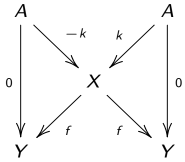

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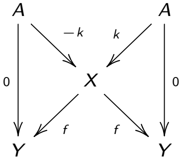

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In particular, we consider butterflies of this kind where domain and codomain are $(ZK \xrightarrow{0} Y, \phi^* \xi)$. We can denote by $\mathrm{H}^2(Y, ZK, \phi^* \xi)$ the abelian group of isomorphism classes of such butterflies.

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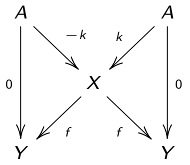

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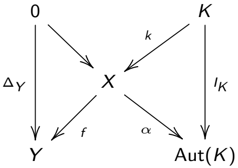
Going back to general extensions



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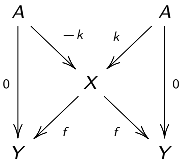

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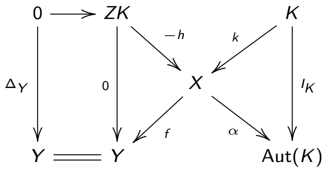

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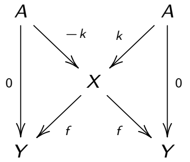

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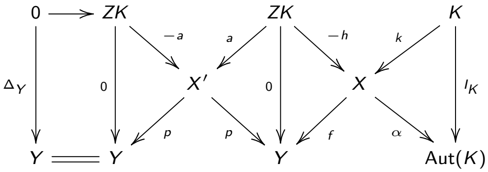

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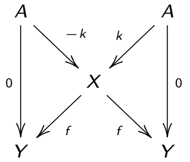

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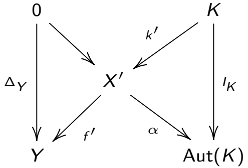

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This is the intrinsic Schreier-Mac Lane theorem [Bourn '08].

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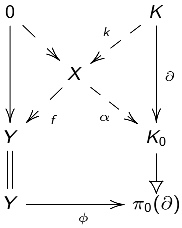

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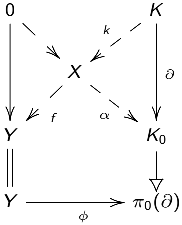

Theorem

Either $\mathrm{Ext}(Y, \partial, \phi)$ *is empty, or it is a simply transitive* $\mathrm{H}^2(Y, \pi_1(\partial), \phi^* \overline{\xi})$*-set.*