A presentation of the book

Schreier split epimorphisms in monoids and in semirings

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Outline

Introduction

Schreier split epimorphisms in monoids

Semirings
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During the last years there has been a great interest in finding a suitable categorical framework to study group-like structures:

- Mal’tsev categories
- protomodular categories
- homological categories
- semi-abelian categories

Some beautiful theories have been developed in these categories: commutators, homology, cohomology, torsion theories, radicals, etc.

These theories have led to a conceptual understanding of parallel results in $\text{Grp}$, $\text{Rng}$, $\text{Lie}_K$, $\text{XMod}$, $\text{Grp(Comp)}$. 
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These theories have led to a conceptual understanding of parallel results in \( \text{Grp}, \text{Rng}, \text{Lie}_K, \text{XMod}, \text{Grp(Comp)} \).
Question
What can be said about the categorical properties of the category Mon of monoids?

Although Mon is not a Mal’tsev category, it is a unital category (Bourn, 1996):

Definition
A finitely complete pointed category $C$ is unital when, given two objects $A$ and $B$ in $C$, the morphisms $(1_A, 0)$ and $(0, 1_B)$ in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{(1_A, 0)} & A \times B \\
\downarrow & & \downarrow \\
& & B \\
& \xleftarrow{(0, 1_B)} & 
\end{array}
$$

are jointly extremal epimorphic.
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What can be said about the **categorical properties** of the category **Mon** of monoids?

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\end{array}
\]

are jointly extremal epimorphic.
This means that, given a monomorphism \( m: M \rightarrow A \times B \)

\[
\begin{array}{c}
M \\
\downarrow^m \\
\end{array}
\quad
\begin{array}{c}
A \\
\rightarrow \\
\downarrow^{(1_A,0)} \\
A \times B \\
\leftarrow \\
\downarrow^{(0,1_B)} \\
B
\end{array}
\]

such that \((1_A, 0)\) and \((0, 1_B)\) factor through \(m\)
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This means that, given a monomorphism \( m: M \to A \times B \)

\[
\begin{array}{c}
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\downarrow (1_A, 0) \\
M \\
\downarrow \Rightarrow \downarrow m \\
A \times B \\
\downarrow \Rightarrow \\
(0, 1_B) \\
B
\end{array}
\]

such that \((1_A, 0)\) and \((0, 1_B)\) factors through \( m \), then \( m \) is an iso.
This implies in particular that the arrows

\[ A \xrightarrow{(1_A,0)} A \times B \xleftarrow{(0,1_B)} B \]

are jointly epimorphic.

This opens the way to the study of commuting arrows:

given two arrows \( a: A \to C \) and \( b: B \to C \) with the same codomain, there is at most one arrow \( \phi \) making the diagram

\[ A \xrightarrow{(1_A,0)} A \times B \xleftarrow{(0,1_B)} B \]

commute.
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\[
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A & \xrightarrow{(1_A,0)} & A \times B & \xleftarrow{(0,1_B)} & B \\
\downarrow a & & \downarrow \phi & & \downarrow b \\
C & & & &
\end{array}
\]

commute.
When this is the case,

\[
\begin{array}{c}
A \xrightarrow{(1_A,0)} A \times B & \xleftarrow{(0,1_B)} B \\
\downarrow^a & \downarrow^\phi & \downarrow^b \\
C & & \\
\end{array}
\]

one says that \textit{a} and \textit{b commute} (in the sense of Huq, 1968).

In the category \textit{Mon} there is a nice theory of commuting arrows, leading to a \textit{commutator theory of subobjects}. 
When this is the case, one says that \( a \) and \( b \) commute (in the sense of Huq, 1968).

In the category \( \text{Mon} \) there is a nice theory of commuting arrows, leading to a commutator theory of subobjects.
Can one develop some other aspects of categorical algebra in \textbf{Mon}?

Is there a structural property of the fibration of points in \textbf{Mon}, as it is the case in the category \textbf{Grp} of groups?

The book \textit{Schreier split epimorphisms in monoids and in semirings} gives a positive and very interesting answer!
Can one develop some other aspects of categorical algebra in Mon?

Is there a structural property of the fibration of points in Mon, as it is the case in the category Grp of groups?

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Can one develop some other aspects of categorical algebra in \textbf{Mon}?

Is there a structural property of the \textit{fibration of points} in \textbf{Mon}, as it is the case in the category \textbf{Grp} of groups?

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Schreier split epimorphisms in monoids

Semirings
Schreier split epimorphisms in monoids

Recall that the fibration of points concerns the category $\text{Pt}(\mathcal{C})$:

- objects: split epimorphisms in $\mathcal{C}$

$$
\begin{array}{ccc}
A & \xrightarrow{p} & B \\
\downarrow{s} & & \downarrow{s'} \\
A' & \xleftarrow{p'} & B'
\end{array}
$$

- morphisms: pairs of arrows $(f_A, f_B)$ in $\mathcal{C}$ making the diagram commute.
Schreier split epimorphisms in monoids

Recall that the fibration of points concerns the category $\text{Pt}(\mathbb{C})$:

- objects: split epimorphisms in $\mathbb{C}$

  \[
  \begin{array}{c}
  A \\ \downarrow p \\
  \leftarrow \quad \rightarrow \\
  B \\
  \end{array}
  \quad ps = 1_B
  \]

- morphisms: pairs of arrows $(f_A, f_B)$ in $\mathbb{C}$ making the diagram commute.

\[
\begin{array}{c}
A \\ \downarrow p \\
\leftarrow \\
B \\
\end{array}
\quad \begin{array}{c}
A' \\ \downarrow p' \\
\leftarrow \\
B' \\
\end{array}
\quad \begin{array}{c}
f_A \\ \downarrow s \\
\leftarrow \\
B \\
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Schreier split epimorphisms in monoids
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\]
There is a functor \( P : \text{Pt}(\mathcal{C}) \to \mathcal{C} \) associating, with any split epimorphism, its codomain:

\[
\begin{array}{ccc}
A & \xrightarrow{p} & B \\
\downarrow{f_A} & & \downarrow{f_B} \\
A' & \xrightarrow{s'} & B'
\end{array}
\]

is sent by \( P \) to

\[
\begin{array}{ccc}
\quad \quad & \quad \quad & \\
\quad \quad & \quad \quad & \\
\quad \quad & \quad \quad & \\
B & \downarrow{f_B} & B'
\end{array}
\]

This functor \( P : \text{Pt}(\mathcal{C}) \to \mathcal{C} \) is called the fibration of pointed objects.
There is a functor $P : \text{Pt}(C) \to C$ associating, with any split epimorphism, its codomain:

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A & \xrightarrow{p} & B \\
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is sent by $P$ to

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\end{array}
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\]

is sent by $P$ to

\[
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B & \xrightarrow{f_B} & B' \\
\downarrow & & \\
B & \to & B'
\end{array}
\]

This functor $P : \text{Pt}(\mathcal{C}) \to \mathcal{C}$ is called the fibration of pointed objects.
One discovery in this book is that, in \textbf{Mon}, one should consider \textbf{SPt(Mon)}, the category of “Schreier split epimorphisms in Mon”: 

let

\[
\begin{array}{c}
0 \rightarrow K \xrightarrow{k} A \xrightarrow{p} B \xrightarrow{s} 0
\end{array}
\]

be a split epi in \textbf{Mon}, with kernel \( k : K \rightarrow A \).

This is a \textbf{Schreier split epi} if, for any \( a \in A \), there is a unique \( k \in K \) such that

\[
a = k \cdot \text{sp}(a).
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This is a Schreier split epi if, for any $a \in A$, there is a unique $k \in K$ such that

\[a = k \cdot sp(a)\].
Remark
Any Schreier split epi in Mon determines a set-theoretic map $q$

\[
0 \rightarrow K \xrightarrow{k} A \xrightarrow{p} B \rightarrow 0
\]

defined by $q(a) = k$, for any $a \in A$, where $k \in K$ is such that

\[
a = k \cdot sp(a).
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The map $q$ is the Schreier retraction associated with the Schreier split exact sequence.
**Remark**

Any *Schreier split epi* in *Mon* determines a set-theoretic map $q$

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The map $q$ is the *Schreier retraction* associated with the *Schreier split exact sequence*. 
Example
The canonical split epi in Mon given by

\[
\begin{array}{c}
0 \rightarrow A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_2} B \rightarrow 0
\end{array}
\]

\[\begin{array}{c}
(1_A,0) \\
(0,1_B)
\end{array}\]

is a Schreier split epi.
Example

Any split epimorphism

\[ 0 \longrightarrow K \xleftarrow{q} A \xrightarrow{p} B \longrightarrow 0 \]

in the category \textbf{Grp} is a Schreier split epi:

indeed, given \( a \in A \), choose \( q(a) = k = a \cdot sp(a)^{-1} \in K \), and

\[ k \cdot sp(a) = (a \cdot sp(a)^{-1}) \cdot sp(a) = a. \]
Example

Any split epimorphism

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & A & \rightarrow & B & \rightarrow & 0 \\
& & ^q & \downarrow & ^p & \leftarrow & ^s & \; \\
& & _k & \; & & \leftarrow & & \\
\end{array}
\]

in the category $\text{Grp}$ is a Schreier split epi:

indeed, given $a \in A$, choose $q(a) = k = a \cdot \text{sp}(a)^{-1} \in K$, and

\[k \cdot \text{sp}(a) = (a \cdot \text{sp}(a)^{-1}) \cdot \text{sp}(a) = a.\]
In the category Mon, the Schreier split epis behave extremely well:

Lemma
Given a Schreier split epimorphism in Mon equipped with its kernel

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \overset{k}{\rightarrow} & A & \overset{p}{\rightarrow} & B \\
\end{array}
\]

then \( p = \text{coker}(k) : \)

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \overset{k}{\rightarrow} & A & \overset{p}{\rightarrow} & B & \rightarrow & 0. \\
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Remark
This is due to the fact that the pair \((k, s)\) is jointly epimorphic.
In the category $\text{Mon}$, the Schreier split epis behave extremely well:

**Lemma**
Given a Schreier split epimorphism in $\text{Mon}$ equipped with its kernel

$$
0 \longrightarrow K \overset{k}{\longrightarrow} A \overset{p}{\longrightarrow} B \\
0 \leftarrow \leftarrow K \overset{k}{\longleftarrow} A \overset{s}{\longleftarrow} B
$$

then $p = \text{coker}(k)$:

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0 \longrightarrow K \overset{k}{\longrightarrow} A \overset{p}{\longrightarrow} B \longrightarrow 0.
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**Remark**

This is due to the fact that the pair $(k, s)$ is jointly epimorphic.
Theorem
Given a commutative diagram of Schreier split exact sequences

in Mon, if \( u \) is an iso then \( v \) is an iso.
An analogy then appears between the situations in \( \text{Grp} \) and in \( \text{Mon} \):

**Groups**
For any \( f : X \to Y \) in \( \text{Grp} \) the change-of-base functor

\[
f^* : \text{Pt}_Y(\text{Grp}) \to \text{Pt}_X(\text{Grp})
\]

with respect to the fibration \( P : \text{Pt}(\text{Grp}) \to \text{Grp} \) is conservative.

**Monoids**
For any \( f : X \to Y \) in \( \text{Mon} \) the change-of-base functor

\[
f^* : \text{SPt}_Y(\text{Mon}) \to \text{SPt}_X(\text{Mon})
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with respect to the fibration \( P^S : \text{SPt}(\text{Mon}) \to \text{Mon} \) is conservative.
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**Monoids**
For any $f: X \to Y$ in $\text{Mon}$ the change-of-base functor

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\textbf{Monoids}
For any \( f : X \to Y \) in \textbf{Mon} the change-of-base functor

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with respect to the fibration \( P^S : \text{SPt}(\text{Mon}) \to \text{Mon} \) is conservative.
The full subcategory $\text{SPt}(\text{Mon})$ of $\text{Pt}(\text{Mon})$ determines a subfibration $P^S$ of the fibration of points $P$:

\[
\begin{array}{ccc}
\text{SPt(Mon)} & \xrightarrow{j} & \text{Pt(Mon)} \\
\downarrow P^S & & \downarrow P \\
\text{Mon} & & \text{Mon}
\end{array}
\]
These observations lead to a detailed study of **internal categorical structures** in **Mon**:

- Schreier internal categories (Patchkoria, 1998),
- Schreier internal groupoids,
- Schreier internal relations,
- centralizers of Schreier reflexive relations.
These observations lead to a detailed study of internal categorical structures in $\text{Mon}$:

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Split extension classifier

In Mon, for any monoid $M$, it is shown that the monoid $\text{End}(M)$ of endomorphisms of $M$ has a universal property, which is analogous to the one of the automorphism group $\text{Aut}(G)$ of a group $G$ in Grp.

Indeed, one can construct a Schreier split extension

$$0 \longrightarrow M \longrightarrow \text{Hol}(M) \longrightarrow \text{End}(M) \longrightarrow 0,$$

with the following universal property:
**Split extension classifier**

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Indeed, one can construct a Schreier split extension

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with the following universal property:
for any Schreier split extension with kernel $M$ in $\text{Mon}$

$$
\begin{array}{ccc}
0 & \longrightarrow & M \\
& \searrow_{k} & \nearrow_{s} \\
& \downarrow \phi & \\
0 & \longrightarrow & \text{Hol}(M)
\end{array}
\quad
\begin{array}{ccc}
M & \longrightarrow & A \\
\downarrow p & & \downarrow s \\
B & \longrightarrow & 0
\end{array}
$$

there is a unique arrow $\phi$ making the following diagram commute:
for any Schreier split extension with kernel $M$ in $\text{Mon}$

$$
0 \rightarrow M \xrightarrow{k} A \xrightarrow{\phi} B \xrightarrow{s} 0,
$$

there is a unique arrow $\phi$ making the following diagram commute:

$$
\begin{array}{c}
0 \rightarrow M \xrightarrow{k} A \xrightarrow{\phi} B \xrightarrow{s} 0 \\
0 \rightarrow M \xrightarrow{k} \text{Hol}(M) \xrightarrow{\phi} \text{End}(M) \xrightarrow{s} 0
\end{array}
$$
For this reason the monoid $\text{End}(M)$ is called the **Schreier split extension classifier of $M$**.

The group $\text{Aut}(M)$ is also shown to have a universal property, and it is called the **homogeneous split extension classifier of $M$**.

These concepts are then used in order to classify what the authors call **special Schreier extensions with abelian kernels**.
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Introduction

Schreier split epimorphisms in monoids

Semirings
Semirings

Many of the interesting results discovered by Manuela Sobral and her collaborators in \textbf{Mon} also have analogous versions in the category \textbf{SRng} of semirings.

\textbf{Definition}
\[(A, +, \cdot, 0)\] is a \textit{semiring} if
\begin{itemize}
  \item \((A, +, 0)\) is a commutative monoid;
  \item \(\cdot : A \times A \to A\) is an associative binary operation such that
    \[a \cdot (b + c) = a \cdot b + a \cdot c\]
    \[(a + b) \cdot c = a \cdot c + b \cdot c.\]
\end{itemize}

\textbf{Fact :}
The category \textbf{SRng} is unital.
Semirings

Many of the interesting results discovered by Manuela Sobral and her collaborators in Mon also have analogous versions in the category SRng of semirings.

**Definition**

$(A, +, \cdot, 0)$ is a **semiring** if

- $(A, +, 0)$ is a commutative **monoid**;
- $\cdot : A \times A \to A$ is an associative binary operation such that

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a \cdot (b + c) = a \cdot b + a \cdot c
\]

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Many of the interesting results discovered by Manuela Sobral and her collaborators in Mon also have analogous versions in the category SRng of semirings.

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  \]
  
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  \]

**Fact:**
The category SRng is unital.
**Definition**

A split epi

\[
0 \rightarrow K \xrightarrow{k} A \xleftarrow{s} B \rightarrow 0
\]

in **SemiRng**, with kernel \( k : K \rightarrow A \), is a **Schreier split epi** if, for any \( a \in A \), there is a unique \( k \in K \) such that

\[
a = k + sp(a).
\]

The fibration

\[
SPt(SemiRng) \rightarrow SemiRng
\]

of Schreier pointed objects in **SemiRng** has some remarkable properties, analogous to the ones of the fibration

\[
P^S : SPt(Mon) \rightarrow Mon
\]
**Definition**

A split epi

\[ 0 \to K \to A \overset{p}{\longrightarrow} B \to 0 \]

in \textit{SemiRng}, with kernel \( k : K \to A \), is a Schreier split epi if, for any \( a \in A \), there is a unique \( k \in K \) such that

\[ a = k + sp(a). \]

The fibration

\[ \text{SPt}(\text{SemiRng}) \to \text{SemiRng} \]

of Schreier pointed objects in \textit{SemiRng} has some remarkable properties, analogous to the ones of the fibration

\[ P^S : \text{SPt}(\text{Mon}) \to \text{Mon} \]
The results established in the semiring case give a structural meaning to the intuitive proportion:

\[ \text{Mon} : \text{Grp} = \text{SRng} : \text{Rng} \]
The results established in the semiring case give a structural meaning to the intuitive proportion:

$$\text{Mon} : \text{Grp} = \text{SRng} : \text{Rng}.$$
The book

Schreier split epimorphisms in monoids and in semirings
by D. Bourn, N. Martins-Ferreira, A. Montoli, and M. Sobral

*Texts in Mathematics of the Department of Mathematics of the University of Coimbra*

sheds some new light on the categories Mon and SemiRng, by providing a categorical foundation to the study of monoids and semirings.
Happy Birthday Manuela!