Graphs, polarities and completions of lattices

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Categorical Methods in Algebra and Topology

Workshop in honour of Manuela Sobral

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Polarities

A **polarity** is a triple (X, Y, R) where X and Y are non-empty sets and $R \subseteq X \times Y$ is a binary relation from X to Y.

Let $\mathbf{L}$ be a bounded lattice and consider

$$\mathcal{F}(\mathbf{L}) = \{\text{filters of } \mathbf{L}\} \quad \text{and} \quad \mathcal{I}(\mathbf{L}) = \{\text{ideals of } \mathbf{L}\}$$

For $R \subseteq \mathcal{F}(\mathbf{L}) \times \mathcal{I}(\mathbf{L})$ defined as follows

$$FRI \iff F \cap I \neq \emptyset,$$

the triple $(\mathcal{F}(\mathbf{L}), \mathcal{I}(\mathbf{L}), R)$ **is a polarity**.

The polarity given by non-empty intersection between the filters and the ideals of $\mathbf{L}$ yields a Galois connection:

$$(\ )^R \colon \mathcal{P}(\mathcal{F}(\mathbf{L})) \rightleftarrows \mathcal{P}(\mathcal{I}(\mathbf{L})) \colon {}^R(\ )$$

where

$$A^R = \{\, I \mid \forall F \in A \quad FRI \,\}$$

and

$${}^R B = \{\, F \mid \forall I \in B \quad FRI \,\}.$$

The set of Galois closed subsets $\mathcal{G}(\mathbf{L}) = \{ U \subseteq \mathcal{F}(\mathbf{L}) \mid U =^R (U^R) \}$ is a complete lattice.

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- For the embedding $e\colon \mathbf{L} \to \mathcal{G}(\mathbf{L})$ defined by $e(a) = \{ F \in (\mathbf{L}) \mid a \in F \}$, $(e, \mathcal{G}(\mathbf{L}))$ is a completion of $\mathbf{L}$.

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$$\bigwedge e(F) \leq \bigvee e(I) \Leftrightarrow F \cap I \neq \emptyset$$

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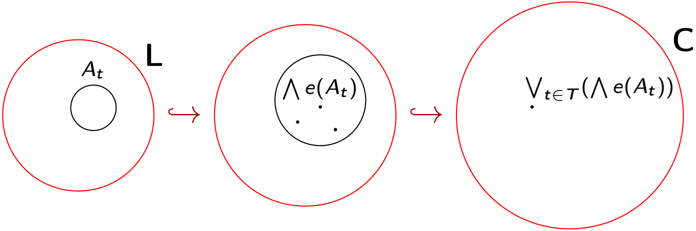

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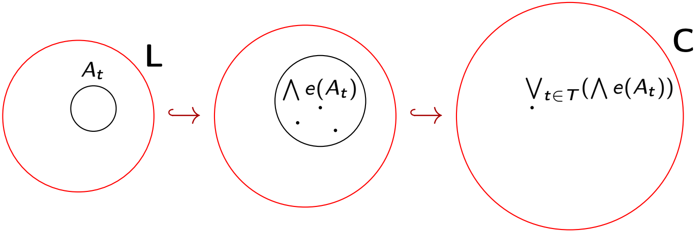

and

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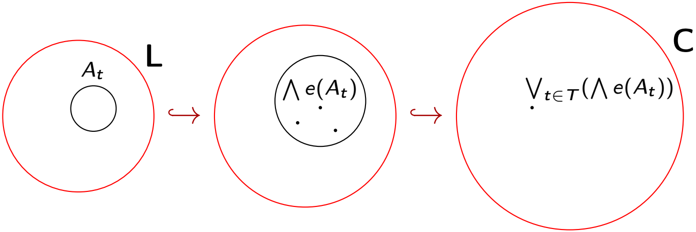

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Canonical extensions, Gehrke & Harding (2001)

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- A **canonical extension** of $\mathbf{L}$ is a **completion** $(e, \mathbf{C})$ of $\mathbf{L}$ that is simultaneously **compact** and **dense**.
- Canonical extensions are unique up to isomorphism.
- Hence the completion $(e, \mathcal{G}(\mathbf{L}))$ of $\mathbf{L}$, or simply $\mathcal{G}(\mathbf{L})$, is the canonical extension of $\mathbf{L}$.

Perfect lattices

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- The canonical extension $\mathcal{G}(\mathbf{L})$ of $\mathbf{L}$ is a perfect lattice.

Perfect lattices and RS frames

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$$\mathbf{L} \;\rightarrow\; (\mathcal{J}^{\infty}(\mathbf{L}), \mathcal{M}^{\infty}(\mathbf{L}), \leq)$$

where

- $\mathcal{J}^{\infty}(\mathbf{L})$ is the set of completely join-irreducible elements of $\mathbf{L}$,
- $\mathcal{M}^{\infty}(\mathbf{L})$ is the set of completely meet-irreducible elements of $\mathbf{L}$
- $\leq$ is the order on $\mathbf{L}$ restricted to $\mathcal{J}^{\infty}(\mathbf{L}) \times \mathcal{M}^{\infty}(\mathbf{L})$.

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  (i) $x_1 \neq x_2$ implies $x_1^R \neq x_2^R$;

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- and reduced, i.e.,

  (i) for every $x \in \mathcal{J}^\infty(\mathbf{L})$ there exists $y \in \mathcal{M}^\infty(\mathbf{L})$ such that $\neg(xRy)$ and $\forall w \in \mathcal{J}^\infty(\mathbf{L})\ ((w \neq x \ \ \& \ \ xR \subseteq wR) \Rightarrow wRy)$;

  (ii) for every $y \in \mathcal{M}^\infty(\mathbf{L})$ there exists $x \in \mathcal{J}^\infty(\mathbf{L})$ such that $\neg(xRy)$ and $\forall z \in \mathcal{M}^\infty(\mathbf{L})\ ((z \neq y \ \ \& \ \ Ry \subseteq Rz) \Rightarrow xRz)$.

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The polarities that are separated and reduced are called **RS frames**.

Perfect lattices and RS frames

(Dunn, Gehrke, Palmigiano (2005), Gehrke(2006))

$$\mathbf{PerLat} \underset{}{\overset{\text{Gehrke}}{\rightleftarrows}} (\mathbf{RS})\mathbf{Fr}$$

$\mathbf{PerLat}$:

category of perfect lattices with complete lattice homomorphisms.

$\mathbf{RSFr}$:

category of RS frames with RS morphisms.

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Maximal filter-ideal pairs and maximal partial homomorphisms

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- A maximal filter-ideal pair of $\mathbf{L}$ is a pair (F, I) such that
  - F is a filter and I is an ideal of $\mathbf{L}$;
  - F is maximal in $\mathcal{F}(\mathbf{L})$ with respect to not intersect I;
  - I is maximal in $\mathcal{I}(\mathbf{L})$ with respect to not intersect F.

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- A partial homomorphism $f : \mathbf{L} \to 2$ is a partial map such that $\operatorname{dom} f$ is a bounded sublattice of **L** and $f\restriction_{\operatorname{dom} f} : \operatorname{dom} f \to 2$ is a bounded lattice homomorphism.

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- A maximal partial homomorphism is a partial homomorphism $f\colon \mathbf{L} \to 2$ which is not properly extended by any partial homomorphism $g\colon \mathbf{L} \to 2$.

Bounded lattices and graphs

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- The dual graph of **L** is the graph $(\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{2}), E)$ where

  (i) the vertex set is the set $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{2})$ of all maximal partial homomorphisms from **L** to 2;

  (ii) the set E is formed by the pairs (f, g) such that $f \leq G$, or equivalently, $f^{-1}(1) \cap g^{-1}(0) = \emptyset$.

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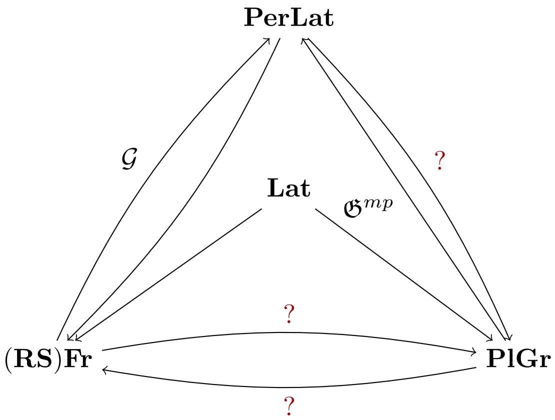
- The lattice $\mathcal{G}^{mp}(\mathbf{X}, \underset{\sim}{2})$ ordered by

$$\varphi \leq \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$$

  is the canonical extension of $\mathbf{L}$.

Graphs, RS frames and canonical extensions



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(i) $\neg(wRz)$;
(ii) $xR \subseteq wR$ and $Ry \subseteq Rz$;
(iii) for every $u \in X_1$, if $u \neq w$ and $wR \subseteq uR$ then uRz;
(iv) for every $v \in X_2$, if $v \neq z$ and $Rz \subseteq Rv$ then wRv.

TiRS Graphs and TiRS frames: an equivalence of categories

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\varphi} & \mathbf{Y} \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{gr}(\rho(\mathbf{X})) & \xrightarrow{\mathrm{gr}(\rho(\varphi))} & \mathrm{gr}(\rho(\mathbf{Y})) \end{array}$$

$\mathbf{X}$ and $\mathbf{Y}$ are TiRS frames

$$\begin{array}{ccc} \mathbf{F} & \xrightarrow{\psi} & \mathbf{G} \\ \downarrow \cong & & \downarrow \cong \\ \rho(\mathrm{gr}(\mathbf{F})) & \xrightarrow{\rho(\mathrm{gr}(\psi))} & \rho(\mathrm{gr}(\mathbf{G})) \end{array}$$

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