Abstract characterisation of varieties and quasivarieties of ordered algebras

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Recollection of Birkhoff's Theorems (1935)

Quasi/varieties as closed subclasses of algebras for a given fixed signature.

Varieties = HSP classes. Quasivarieties = SP classes.

Recognition Theorems (Linton/Lawvere/Duskin. . . 1960's)

Quasi/varieties are abstract categories with certain properties.

Characterisations essentially of the form:

> A category $\mathscr{A}$ is equivalent to a quasivariety/variety of finitary one-sorted algebras iff $\mathscr{A}$ is regular/exact, cocomplete, and has a nice generator.[a]

[a] I.e., an object that pretends to be a free algebra on one generator.

What is regularity and exactness, roughly?

Regularity: congruences correspond to quotients.
Exactness: regularity + all congruences are nice.

Why do recognition theorems hold?

The base category Set is exact (and therefore regular).

1. Regularity of Set: surjections correspond to equivalence relations.
2. Exactness of Set: every equivalence relation has the form $\{(x', x) \mid f(x') = f(x)\}$ for a suitable mapping f.

More details in:

M. Barr, P. A. Grillet, D. H. van Osdol, *Exact categories and categories of sheaves*, LNM 236, Springer 1971

The goal: Recognition theorems for ordered algebras

We want to characterise quasi/varieties of ordered algebras as abstract categories.

A plethora of problems in the ordered world

1. What do we mean by an ordered algebra?
2. What are quasi/varieties of ordered algebras?
3. Are there Birkhoff-type theorems?
4. Can one use ordinary regularity and exactness?

   NO: The (ordinary) category of posets and monotone mappings is not exact (in the sense of M. Barr).
5. What are abstract congruences in the ordered world?

Example (Kleene algebras)

A Kleene algebra A consists of a poset $(A_0, \leq)$, together with monotone operations

$$+, \cdot : (A_0, \leq) \times (A_0, \leq) \to (A_0, \leq), \quad 0, 1 : \mathbb{1} \to (A_0, \leq),$$
$$(-)^* : (A_0, \leq) \to (A_0, \leq)$$

subject to axioms that $((A_0, \leq), 0, 1, +, \cdot)$ is an ordered semiring and such that[a]

$$x + x = x, \quad 1 + x(x^*) \leq x^*, \quad 1 + (x^*)x \leq x^*,$$
$$yx \leq x \Rightarrow (y^*)x \leq x, \quad xy \leq x \Rightarrow x(y^*) \leq x$$

holds.

Homomorphisms are monotone maps preserving the operations.

[a] Intuition: $x^* = \sum_{i=0}^{\infty} x^i$, had such infinite sums existed.

Example (nice, but quite disturbing)

A set A is a poset $(A_0, \leq)$ together with no operations subject to axiom

$$x \leq y \Rightarrow y \leq x$$

Homomorphisms are monotone maps preserving the operations.

1. By the above, sets seem to form an ordered quasivariety.
2. But: sets seem to form an ordered variety if "strange" arities are allowed:

$$\Sigma 2 = \{\sigma_0 \leq \sigma_1\}$$

Here 2 is the two-element chain.

Indeed, consider the equalities:

$$\sigma_0(x, y) = y, \quad \sigma_1(x, y) = x$$

We restrict ourselves to the easier situation

1. The base category for ordered algebras: the category Pos of all posets and all monotone maps.
2. We pass from ordinary categories and functors to category theory enriched over Pos.
    1. $\mathscr{X}$ a category = hom-sets are posets, composition is monotone.
    2. $F : \mathscr{X} \to \mathscr{Y}$ a functor = it is a locally monotone functor (the action on arrows is monotone).
3. Nice signatures that have only operations of nice arities: a bounded signature is a functor $\Sigma : |\mathsf{Set}_\lambda| \to \mathsf{Pos}$, where λ is a regular cardinal.
   Here, Σn is the poset of all n-ary operations, $n < \lambda$.

Algebras and homomorphisms

An ordered algebra for Σ is a poset A, together with a monotone map $[\![\sigma]\!] : A^n \to A$, for every σ in Σn, $n < \lambda$.
Moreover, $[\![\sigma]\!] \leq [\![\tau]\!]$ holds pointwise, whenever $\sigma \leq \tau$ in the poset Σn.

A homomorphism from $(A, [\![-]\!])$ to $(B, [\![-]\!])$ is a monotone map $h : A \to B$ such that $h([\![\sigma]\!](a_i)) = [\![\sigma]\!](h(a_i))$ holds for all σ in Σn.

The category of ordered algebras and homomorphisms

All algebras for Σ and all homorphisms form a category $\mathsf{Alg}(\Sigma)$.

There is a (locally monotone) functor $U : \mathsf{Alg}(\Sigma) \to \mathsf{Pos}$.

Ordered quasi/varieties (Steve Bloom & Jesse Wright)

An (enriched) category $\mathscr{A}$, equivalent to a full subcategory of $\mathrm{Alg}(\Sigma)$, spanned by algebras satisfying inequalities of the form

$$s(x_i) \sqsubseteq t(y_j)$$

is called an ordered variety.

If $\mathscr{A}$ is equivalent to a full subcategory of $\mathrm{Alg}(\Sigma)$, spanned by algebras satisfying inequality-implications of the form

$$(\bigwedge_j s_j(x_{ji}) \sqsubseteq t_j(y_{ji})) \Rightarrow s(x_i) \sqsubseteq t(y_j)$$

then it is called an ordered quasivariety.

Steve Bloom & Jesse Wright, 1976 and 1983

$\mathscr{A}$ is an ordered variety iff it is an HSP-class in $\mathsf{Alg}(\Sigma)$.

$\mathscr{A}$ is an ordered quasivariety iff it is an SP-class in $\mathsf{Alg}(\Sigma)$.

Notice: H means "monotone surjections", S means "monotone maps reflecting the order", P means "order-enriched products".

Main results

1. $\mathscr{A}$ is an ordered variety iff it is exact, cocomplete and has a nice generator.[a]
2. $\mathscr{A}$ is an ordered quasivariety iff it is regular, cocomplete and has a nice generator.[a]
3. $\mathscr{A}$ is equivalent to a variety of one-sorted finitary algebras iif $\mathscr{A} \simeq \mathsf{Pos}^{\mathbb{T}}$ for a strongly finitary[b] monad $\mathbb{T}$ on Pos. Moreover: $\mathsf{Th}(\mathbb{T}) \to \mathsf{Pos}^{\mathbb{T}}$ is a free cocompletion under sifted colimits, where $\mathsf{Th}(\mathbb{T})$ — the theory of $\mathbb{T}$ — is the full subactegory of $\mathsf{Kl}(\mathbb{T})$ spanned by free algebras on finite discrete posets.

Regularity & exactness must be taken in the enriched sense.

[a] In the one-sorted case: an object that pretends to be a free algebra on one generator.

[b] Strongly finitary = preserves (enriched) sifted colimits. A sifted colimit is one weighted by a sifted weight.

Convention

All categories, functors, etc. from now on are enriched in the symmetric monoidal closed category Pos of posets and monotone maps.[a]

[a]Analogous notions/results can be stated for the enrichment in Cat — this is essentially only more technical. But it certainly yields more applications.

Regularity and exactness of a category $\mathscr{X}$

We need:

1. Finite (weighted) limits in $\mathscr{X}$.[a]
2. A good factorisation $(\mathcal{E}, \mathcal{M})$ system in $\mathscr{X}$.
3. A notion of a congruence and its quotient.

[a]A standard reference is: G. M. Kelly, Structures defined by finite limits in the enriched context I, *Cahiers de Top. et Géom. Diff.* XXIII.1 (1982), 3–42.

The factorisation system

1. The "monos": Say $m : X \to Y$ in $\mathscr{X}$ is order-reflecting (it is in $\mathcal{M}$), if the monotone map

$$\mathscr{X}(Z, m) : \mathscr{X}(Z, X) \to \mathscr{X}(Z, Y)$$

reflects orders in Pos.

Hence, $m : X \to Y$ has to satisfy:

$$m \cdot x \leq m \cdot y \text{ in } \mathscr{X}(Z, Y) \quad \text{implies} \quad x \leq y \text{ in } \mathscr{X}(Z, X)$$

for every $x, y : Z \to X$.

2. The "epis" (members of $\mathcal{E}$): via diagonalisation. They are called surjective on objects.

Congruences: a very rough idea

Replace $=$ in

$$X_1 = \{(x', x) \mid f(x') = f(x)\}$$

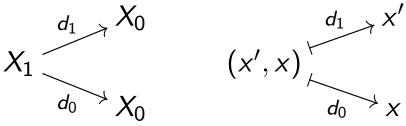
where $f : X_0 \to Z$ is a map, by $\leq$ to obtain

$$X_1 = \{(x', x) \mid f(x') \leq f(x)\}$$

where $f : X_0 \to Z$ is a monotone map.

This could work nicely for "kernels" of monotone maps. What are the abstract properties of X_1?

Most certainly, we are dealing with spans



of monotone maps.

A somewhat better intuition behind a congruence

In a congruence on X_0, one deals with formal squares of the form

$$\begin{array}{ccc} x' & \longrightarrow\!\!\!\!\!\!\!\!\!\!/\;\;\;\; & x \\ \downarrow & & \downarrow \\ y' & \longrightarrow\!\!\!\!\!\!\!\!\!\!/\;\;\;\; & y \end{array}$$

where:

1. The vertices are "objects" of X_0.
2. The horizontal arrows are "specified inequalities": objects of X_1.
3. The vertical arrows are "existing inequalities" in X_0: they give the order in X_1.
4. The specified and existing inequalities interact nicely: "path-lifting property" (discrete fibration in $\mathscr{X}$).
5. The squares can be pasted both horizontally and vertically with no ambiguity (category object in $\mathscr{X}$).

Definition

A congruence in $\mathscr{X}$ is a diagram

$$X_2 \underset{d_0^2}{\overset{d_2^2}{\longrightarrow}} \xrightarrow{d_1^2} X_1 \underset{d_0^1}{\overset{d_1^1}{\longrightarrow}} \xleftarrow{i_0^0} X_0$$

such that

1. It is an internal category in $\mathscr{X}$.
2. The span (d_0^1, X_1, d_1^1) is a two-sided discrete fibration.
3. The morphism $\langle d_0^1, d_1^1 \rangle : X_1 \to X_0 \times X_0$ is an $\mathcal{M}$-morphism.

The quotient of the above congruence is a coinserter $q : X_0 \to Q$ of the pair d_0^1, d_1^1.

The intuition behind a quotient

Given a congruence on X_0, the coinserter of d_0^1 and d_1^1 imposes inequalities of the form

$$a' = a_0 \longrightarrow a_1 \nrightarrow a_2 \longrightarrow \dots \longrightarrow a_{n-2} \nrightarrow a_{n-1} \longrightarrow a_n = a$$

Each of them has an unambiguous form

$$a' \longrightarrow\!\!\!\!\!\!\!\!/\;\;\;\; a$$

since a congruence is a two-sided discrete fibration and an internal category.

This allows proving that

1. In Pos, every congruence has the form $\ker(f)$.
2. In Set, there are congruences not of the form $\ker(f)$.

Definiton (goes back to R. Street 1982)

A category $\mathscr{X}$ is called regular, if

1. $\mathscr{X}$ has finite limits.
2. $\mathscr{X}$ has $(\mathcal{E}, \mathcal{M})$-factorisations.
3. The $\mathcal{E}$-morphisms are stable under pullback.
4. $\mathscr{X}$ has quotients of congruences.

If, in addition, congruences are effective[a] in $\mathscr{X}$, then $\mathscr{X}$ is called exact.

[a] I.e., every congruence has the form $\ker(f)$, where $\ker(f)$ denotes the higher kernel of $f : X \to Y$ in $\mathscr{X}$.

Recent results (R. Garner and J. Bourke)

Regularity and exactness can also be captured by kernel-quotient systems in enriched category theory.

Examples

1. Set is regular but not exact. Hence Set cannot be an ordered variety in any signature.
2. Every "presheaf" category $[\mathscr{S}^{op}, \mathsf{Pos}]$ is exact.

   This includes $[\mathsf{Pos}_{fp}, \mathsf{Pos}]$, i.e., finitary endofunctors of Pos. This fact yields a good behaviour of inequational presentations of finitary endofunctors of Pos. This is important for relation lifting in coalgebraic logic.
3. The category $\mathsf{Mnd}_{strfin}(\mathsf{Pos})$ of strongly finitary monads on Pos is a (many-sorted) variety of ordered algebras.

   This is important for "universal algebra over posets in the clone form".

References

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