A characterisation of R_1-spaces via approximate Mal'tsev operations

Thomas Weighill

Supported by MIH Media Lab and
South African National Research Foundation

Stellenbosch University

Workshop on Categorical Methods in Algebra and Topology
Coimbra 2014

Mal'tsev varieties and categories

Theorem (Mal'tsev, 1954)

For a variety $\mathbb{X}$ of universal algebras, the following are equivalent:

- *the composition of congruences on any object in $\mathbb{X}$ is commutative*
- *the algebraic theory of $\mathbb{X}$ contains a ternary term μ satisfying*

$$\mu(x, y, y) = x = \mu(y, y, x)$$

Mal'tsev varieties and categories

Theorem (Mal'tsev, 1954)

For a variety $\mathbb{X}$ of universal algebras, the following are equivalent:

- *the composition of congruences on any object in $\mathbb{X}$ is commutative*
- *the algebraic theory of $\mathbb{X}$ contains a ternary term μ satisfying*

$$\mu(x, y, y) = x = \mu(y, y, x)$$

- A regular category is a *Mal'tsev category* if composition of equivalence relations is commutative. (A. Carboni, J. Lambeck and M. C. Pedicchio, 1990).

Naturally Mal'tsev categories

- What about internal Mal'tsev operations in a category $\mathbb{X}$?

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\ \mu\ } & X \\ {\scriptstyle(\pi_1,\pi_2,\pi_2)}\uparrow\ \uparrow{\scriptstyle(\pi_2,\pi_2,\pi_1)} & & \Vert \\ X \times X & \xrightarrow[\ \pi_1\ ]{} & X \end{array}$$

Naturally Mal'tsev categories

- What about internal Mal'tsev operations in a category $\mathbb{X}$?

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\ \ \mu\ \ } & X \\ {\scriptstyle (\pi_1,\pi_2,\pi_2)}\uparrow \ \uparrow{\scriptstyle (\pi_2,\pi_2,\pi_1)} & & \Vert \\ X \times X & \xrightarrow[\ \ \pi_1\ \ ]{} & X \end{array}$$

- A *naturally Mal'tsev category* (P. T. Johnstone, 1989) is a category $\mathbb{X}$ where the identity functor $1_{\mathbb{X}}$ admits an internal Mal'tsev operation μ in the functor category $\mathbb{X}^{\mathbb{X}}$.

Naturally Mal'tsev categories

- What about internal Mal'tsev operations in a category $\mathbb{X}$?

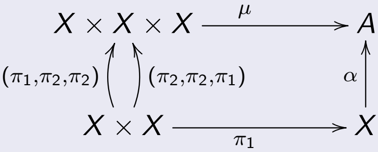
$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\ \mu\ } & X \\ {\scriptstyle (\pi_1,\pi_2,\pi_2)}\uparrow \ \uparrow{\scriptstyle (\pi_2,\pi_2,\pi_1)} & & \| \\ X \times X & \xrightarrow[\ \pi_1\ ]{} & X \end{array}$$

- A *naturally Mal'tsev category* (P. T. Johnstone, 1989) is a category $\mathbb{X}$ where the identity functor $1_{\mathbb{X}}$ admits an internal Mal'tsev operation μ in the functor category $\mathbb{X}^{\mathbb{X}}$.
- This turns out to be too strong (for example, the category of groups is not a naturally Mal'tsev category).

Approximate Mal'tsev operations

Definition (D. Bourn and Z. Janelidze, 2008)

In a category $\mathbb{C}$, a morphism $\mu : X^3 \to A$ is an *approximate Mal'tsev operation with approximation* $\alpha : X \to A$ if the following diagram commutes:



Approximate Mal'tsev co-operations

- The dual notion is that of an approximate Mal'tsev co-operation:

Approximate Mal'tsev co-operations

- The dual notion is that of an approximate Mal'tsev co-operation:

$$
\begin{array}{ccc}
X + X + X & \xleftarrow[\mu]{} & A \\
{\scriptstyle (\iota_2,\iota_2,\iota_1)} \downarrow\;\downarrow {\scriptstyle (\iota_1,\iota_2,\iota_2)} & & \downarrow {\scriptstyle \alpha} \\
X + X & \xleftarrow{\iota_1} & X
\end{array}
$$

Approximate Mal'tsev co-operations

- The dual notion is that of an approximate Mal'tsev co-operation:

$$\begin{array}{ccc} X+X+X & \xleftarrow{\mu} & A \\ {\scriptstyle (\iota_2,\iota_2,\iota_1)}\downarrow\;\downarrow{\scriptstyle (\iota_1,\iota_2,\iota_2)} & & \downarrow{\scriptstyle \alpha} \\ X+X & \xleftarrow{\iota_1} & X \end{array}$$

- D. Bourn and Z. Janelidze proved two characterisations of Mal'tsev categories in terms of approximate Mal'tsev (co-)operations.

Characterisation of Mal'stev categories

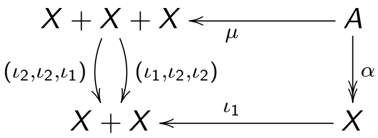
Theorem (D. Bourn and Z Janelidze, 2008)

For a regular category $\mathbb{X}$ with binary coproducts, the following are equivalent:

- $\mathbb{X}$ *is a Mal'tsev category*
- *there exists an approximate Mal'tsev co-operation on* $1_{\mathbb{X}}$ *in the functor category* $\mathbb{X}^{\mathbb{X}}$ *whose approximation* α *has every component a regular epimorphism.*

Characterisation of Mal'stev categories

In other words, every object X is part of the commutative diagram below, with α a regular epi.

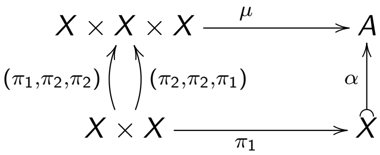


Topological spaces

- The dual of the category of topological spaces, $\mathbf{Top}^{\mathrm{op}}$, is a regular category with binary coproducts, and regular epimorphisms there are precisely the embeddings of topological spaces.

Topological spaces

- The dual of the category of topological spaces, $\mathbf{Top}^{\mathrm{op}}$, is a regular category with binary coproducts, and regular epimorphisms there are precisely the embeddings of topological spaces.
- However, not every object in **Top** admits an approximate Mal'tsev operation with α an embedding:



Topological spaces

- The dual of the category of topological spaces, $\mathbf{Top}^{\mathrm{op}}$, is a regular category with binary coproducts, and regular epimorphisms there are precisely the embeddings of topological spaces.
- However, not every object in **Top** admits an approximate Mal'tsev operation with α an embedding:

$$
\begin{array}{ccc}
X \times X \times X & \xrightarrow{\quad \mu \quad} & A \\
(\pi_1,\pi_2,\pi_2) \uparrow \; \uparrow (\pi_2,\pi_2,\pi_1) & & \uparrow \alpha \\
X \times X & \xrightarrow[\quad \pi_1 \quad]{} & X
\end{array}
$$

- Thus $\mathbf{Top}^{\mathrm{op}}$ is not a Mal'tsev category.

Main result

Theorem

In the category of topological spaces, an object X admits an approximate Mal'tsev operation μ with approximation α a regular monomorphism if and only if it is an R_1-space, i.e. it satisfies the following condition:

(1) *For any two points x,y in X, if there exists an open set A such that $x \in A$ but $y \notin A$, then there exist disjoint open sets B and C such that $x \in B$ and $y \in C$.*

Proof

Firstly, it is enough to consider the *universal approximate Mal'tsev operation* on an object X,

Proof

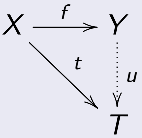
Firstly, it is enough to consider the *universal approximate Mal'tsev operation* on an object X, i.e. μ and α in the diagram below, where C is the colimit of the diagram:

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\mu} & C \\ {\scriptstyle (\pi_1,\pi_2,\pi_2)} \uparrow \; \uparrow {\scriptstyle (\pi_2,\pi_2,\pi_1)} & & \uparrow {\scriptstyle \alpha} \\ X \times X & \xrightarrow[\pi_1]{} & X \end{array}$$

Proof

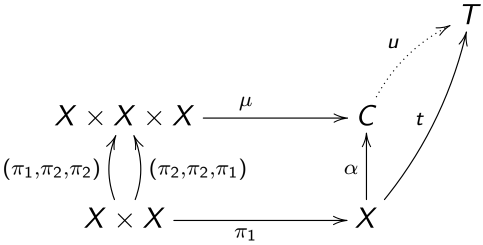
Lemma

A monomorphism $f : X \to Y$ in **Top** is an embedding if and only if for every diagram of solid arrows below, there exists an arrow u making the diagram commute (it is not necessarily unique), where T is the Sierpinski space, i.e. the space T whose underlying set is $\{0, 1\}$ and open sets are $\{\emptyset, T, \{1\}\}$.



Proof

(It is easy to check that α is a monomorphism)



Proof

Required to prove

(1) X is an R_1-space.

(2) For every open set A in X, there exists an open set A' in X^3 which satisfies the following condition:

$$x \in A \Leftrightarrow \forall_{y \in X}(x, y, y) \in A' \Leftrightarrow \forall_{y \in X}(y, y, x) \in A'$$

Proof

$$A' = A^3 \cup \left(\bigcup_{x \in A, y \notin A} (A \cap B_{(x,y)}) \times C_{(x,y)} \times C_{(x,y)} \right) \cup \left(\bigcup_{x \in A, y \notin A} C_{(x,y)} \times C_{(x,y)} \times (A \cap B_{(x,y)}) \right)$$

Proof

Required to prove

(1) X is an R_1-space.

(2) For every open set A in X, there exists an open set A' in X^3 which satisfies the following condition:

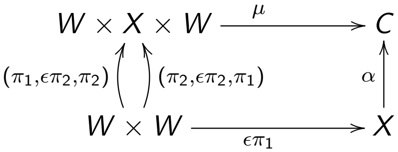
$$x \in A \Leftrightarrow \forall_{y \in X}(x, y, y) \in A' \Leftrightarrow \forall_{y \in X}(y, y, x) \in A'$$

Concluding remarks

- It would be interesting to see what other conditions arising from algebra have duals which which are well-known conditions in topological spaces.

Concluding remarks

- It would be interesting to see what other conditions arising from algebra have duals which which are well-known conditions in topological spaces.
- If we replace the diagram of an approximate Mal'tsev operation with the one below, for an epimorphism $\epsilon : W \to X$, we can characterise R_0 spaces in **Top**:



Thank you.