Workshop on Dualities

University of Coimbra, Portugal

September 19-21, 2016

http://www.mat.uc.pt/~wdual/

Programme

Centre for Mathematics
University of Coimbra

Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR

DMat, Room 2.4

	MONDAY 19
9:00-9:30	Registration
9:30-11:00	Marcel Erné
11:00-11:30	Coffee Break
11:30-12:15	Jorge Picado
12:15-13:00	Pedro Resende

	TUESDAY 20
9:00-10:30	Vincenzo Marra
10:30-11:00	Lurdes Sousa
11:00-11:30	Coffee Break
11:30-12:00	Robert Furber
12:00-12:30	Björn Gohla
12:30-13:00	Juan Pablo Quijano

	WEDNESDAY 21
9:00-10:30	Jean-Eric Pin
10:30-11:15	Maria Manuel Clementino
11:15-11:45	Coffee Break
11:45-13:15	Dirk Hofmann

Lax algebras as spaces: some topological properties and their duals

Maria Manuel Clementino

CMUC, Universidade de Coimbra, Portugal

As pointed out by Janelidze and Sobral in [5], *open* and *perfect* maps can be seen as duals (see also [1]). For finite spaces this extends to *étale* versus *perfect*, but it is no longer valid in general, as shown in [2]. Having this in mind, in this talk we will start by explaining how these notions can be studied in categories enriched in a quantale (see [6]), like in ordered sets, metric spaces or probabilistic metric spaces, or, more generally, in generalised quantale-enriched categories – *lax algebras* – like in topological spaces or approach spaces. This will lead us to the study of other type of duality, between *normal* and *extremally disconnected* spaces (see [3]). This seems to have no relation with the approach of Gutiérrez-García and Picado [4] (and presented by Picado in this workshop), although both approaches emphasise interesting dual facets of normality and extremal disconnectedness.

[1] M.M. Clementino, D. Hofmann, *Triquotient maps via ultrafilter convergence.* Proc. Amer. Math. Soc. 130 (2002), 3423–3431.

[2] M.M. Clementino, D. Hofmann, G. Janelidze, *Local homeomorphisms via ultrafilter convergence.* Proc. Amer. Math. Soc. 133 (2005), 917–922.

[3] E. Colebunders, M.M. Clementino, W. Tholen, *Lax algebras as spaces.* In: Monoidal topology, pp. 375–465, Encyclopedia Math. Appl., 153, Cambridge Univ. Press, Cambridge, 2014.

[4] J. Gutiérrez-García, J. Picado, *On the parallel between normality and extremal disconnectedness.* J. Pure Appl. Algebra 218 (2014), 784–803.

[5] G. Janelidze, M. Sobral, *Finite preorders and topological descent, I.* J. Pure Appl. Algebra 175 (2002), 187–205.

[6] F. W. Lawvere, *Metric spaces, generalized logic, and closed categories.* Rend. Sem. Mat. Fis. Milano 43 (1973), 135–166.

Dualities equivalent to the Ultrafilter Theorem

Marcel Erné

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Leibniz Universität Hannover, Germany

Since the Axiom of Choice (AC) has many desired but also some rather monstrous consequences (for example, Banach's Paradox), it is of interest to derive important mathematical facts in a choice-free manner or at least from weaker choice principles. Perhaps the most frequently used principle in that area of research is the Ultrafilter Theorem or Ultrafilter Principle (UP), requiring that every proper set-theoretical filter be contained in an ultrafilter. Many order-theoretical, topological and algebraic theorems turned out to be equivalent to UP in ZF or NBG set theory. UP is considerably weaker than AC; for example, UP does not even imply the Principle of Dependent Choices, nor conversely, and both principles together are weaker than AC.

We discuss several topological theorems concerning compactness that are equivalent to UP – the most prominent one being Tychonoff's Theorem for products of sober spaces, while that theorem for arbitrary spaces is equivalent to the full Axiom of Choice. One crucial theorem in that context says that every sober space is strictly sober, that is, each open set containing the intersection of a Scott-open filter of open sets contains one of them.

Using these equivalences, one finds that quite diverse dualities relating categories of topological spaces with certain categories of ordered sets or lattices are also equivalent to UP. The classical case is here Stone's duality between Boolean algebras or bounded distributive lattices and compact zero-dimensional spaces or spectral spaces, respectively.

The well-known duality between sober spaces and spatial lattices (complete lattices in which every element is a meet of primes) holds without any choice principles, and the same holds for several important subdualities of that duality, for example, between

(1) strictly spatial frames and strictly sober spaces,
(2) strictly continuous strictly spatial frames and strictly sober locally compact spaces,
(3) algebraic strictly spatial frames and compactly based sober spaces,
(4) coherent strictly spatial frames and spectral spaces.

(A spatial lattice is strictly spatial if each element outside a Scott-open filter is dominated by a prime element outside that filter.)

But some other famous subdualities turn out to be equivalent to UP, for example the Hofmann-Lawson duality between strictly continuous frames and sober locally compact spaces.

Duality for effect algebras and convex sets

Robert Furber

Aalborg University, Denmark

Effect algebras are a generalization of Boolean algebras and MV-algebras intended to include examples such as the lattice of closed subspaces of a Hilbert space or the effects in a C^*-algebra. With Boolean algebras, one can produce Stone duality by using the object 2, being both a Boolean algebra and a Stone space, as a dualizing object. For effect algebras, one can use the real unit interval $[0, 1]$ as a dualizing object, producing a dual adjunction between effect algebras and abstract convex sets. Every adjunction contains an equivalence by restricting to those objects on which the unit and counit are isomorphisms. We can characterize this equivalence using two different kinds of structured ordered Banach space, base-norm spaces and order-unit spaces. The effect algebras involved are the unit intervals of reflexive order-unit spaces, and the convex sets are bases of reflexive base-norm spaces. We can generalize this to two dualities by using the weak-* topology on the dual space instead of requiring the space to be reflexive.

Poincaré Duality as duality of categories

Björn Gohla

Universidade de Lisboa, Portugal

We give a construction that associates a small category $\mathcal{C}(X)$ to a CW-complex X. We obtain interesting families of finite categories from spheres and projective spaces as examples. Under some conditions this category $\mathcal{C}(X)$ seems to represent the homotopy type of X. Interestingly, for finite dimensional X the Poincaré dual $\hat{X}$ has associated to it the opposite category $(\mathcal{C}(X))^{\mathrm{op}} = \mathcal{C}(\hat{X})$.

This is part of a joint project with Benjamin Heredia.

Duality in topology from the point of view of triples

Dirk Hofmann

CIDMA, Department of Mathematics, Universidade de Aveiro, Portugal

The title of this talk is clearly reminiscent to [3] where the author derives the classical duality theorems of Gelfand and Pontrjagin "employing the theory of triples and using only a minimum of analytic information". Whereby Negrepontis' account is based on the Eilenberg–Moore construction, in this talk we wish to show how various duality results appear as restrictions of dualities for Kleisli categories of appropriate monads. More in detail, we will

1. review some general results about (dual) adjunctions and monads, in particular about the structure and construction of dual adjunctions;
2. explain enriched variations of the classical Vietoris monad (see [1]);
3. present duality results involving the Kleisli category of such a monad on ordered or metric compact Hausdorff spaces (see [2]);
4. use our results to obtain dualities for categories of coalgebras for Vietoris functors.

Parts of this talk are based on joint work with Pedro Nora and Renato Neves.

[1] Dirk Hofmann, *The enriched Vietoris monad on representable spaces.* Journal of Pure and Applied Algebra, 218(12):2274–2318, December 2014, `arXiv:1212.5539 [math.CT]`.

[2] Dirk Hofmann and Pedro Nora, *Enriched Stone-type dualities.* Technical report, April 2016, `arXiv:1605.00081 [math.CT]`.

[3] Joan W. Negrepontis, *Duality in analysis from the point of view of triples.* Journal of Algebra, 19(2):228–253, October 1971.

From concrete dual adjunctions to affine adjunctions

Vincenzo Marra

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A collective mathematical effort spanning over five decades has led to a satisfactory general understanding of dual adjunctions between concrete categories. As recognised early on, a pivotal role is played by dualising objects – or, as J. Isbell put it, objects "keeping summer and winter homes". A perspicuous account of the theory was obtained in the late Eighties and early Nineties in papers by W. Tholen and G. Dimov, and by W. Tholen and H. Porst.

One prototypical example of concrete dual adjunction is the one between finitely presented k-algebras and affine varieties in classical algebraic geometry. Passage to the induced duality between nilpotent-free finitely presented k-algebras and affine varieties amounts to Hilbert's Nullstellensatz. One can abstract this and other significant examples into a theoretical framework quite distinct from the one about concrete dual adjunctions. In such an abstraction, it is the affine nature of the "geometric" objects that takes center stage. Beginning with a paper in 1999, Y. Diers developed along these lines a theory of "affine sets" relative to an algebraic theory.

In this talk I present joint work with O. Caramello and L. Spada in which we develop a general theory of affine adjunctions. If the categories at hand are concrete, one essentially recovers the Tholen et al. framework. If the setting is specialised to algebraic theories, the result bears resemblance to Diers' theory, though significant differences remain. I show that a Nullstellensatz-type theorem obtains in abstract form when specialising an affine adjunction to its induced duality; and that in the special case of algebraic theories, the theorem yields non-obvious algebraic information. If time allows I address the original motivations from algebra, geometry, and logic that led to the development of this theory, and discuss open problems.

Hausdorff invariance type theorems for localic maps and their duals

Jorge Picado

CMUC, Universidade de Coimbra, Portugal

Hausdorff was the first to observe (in 1935) that normality is an invariant of closed continuous mappings. In this talk, we use the covariant description of localic maps and sublocales of [J. Picado and A. Pultr, *Frames and Locales: topology without points*, Springer, Basel, 2012] to extend Hausdorff invariance theorem to the pointfree setting. With the introduction of a relative notion of normality, formulated in terms of a fixed class $\mathscr{B}$ of complemented sublocales of the given locale, our approach gains two additional features:

(1) it covers classical variants of normality at once;

(2) just by taking complements in $\mathscr{B}$, it yields dual results about extremal disconnectedness type properties for free.

Other cases treated are e.g. the perfect and hereditary notions.

This is joint work with J. Gutiérrez García (Bilbao) and T. Kubiak (Poznán).

Pervin spaces, another approach to duality

Jean-Éric Pin

IRIF, CNRS and Université Paris-Diderot, France

This lecture is based on a joint work with M. Gehrke and S. Grigorieff (2010) in which topological tools were used to classify formal languages. A Pervin space (a very special case of quasi-uniform space) is simply a set equipped with a lattice of subsets. This notion suffices to define a natural notion of completion and the dual of a lattice of languages is now precisely equal to this completion. Further examples and applications will be given.

Effective equivalence relations and principal quantales

Juan Pablo Quijano*

IST, Universidade de Lisboa, Portugal

In this talk I will describe a generalization of supported quantales which applies to non-unital quantales and to open groupoids beyond étale groupoids [2, 3]. A notion of principal quantal will be introduce which, in the case of groupoid quantales, corresponds precisely to effective equivalence relations. More over, this notion of principal quantal provides a first approach to describe the quantales of étale-complete groupoids [1].

[1] A. Kock and I. Moerdijk, *Presentations of étendues.* Cahiers Topologie Géom. Différentielle Catég. 32 (1991), no. 2, 145-164 (English, with French summary). MR1142688 (92m:18007)

[2] M. C. Protin and P. Resende, *Quantales of open groupoids.* J. Noncommut. Geom. 6 (2012), no. 2, 199-247, DOI 10.4171/JNCG/90. MR2914865

[3] P. Resende, *Étale groupoids and their quantales.* Adv. Math. 208 (2007), no. 1, 147-209. MR2304314 (2008c:22002)

*Joint work with Pedro Resende.

Linear structures on locales

Pedro Resende

IST, Universidade de Lisboa, Portugal

In this talk, based on joint work with João Paulo Santos, I will describe an extension of the adjunction between the category of topological spaces **Top** and the category of locales **Loc** whereby **Top** is replaced by a suitable category of vector bundles on topological spaces and **Loc** is replaced by the so-called category of linearized locales. By a linearized locale is meant a locale L together with a topological vector space A and a meet-preserving map $g\colon L \to \mathrm{Sub}(A)$, where $\mathrm{Sub}(A)$ is the complete lattice of linear subspaces of A, such that the restriction of g to the prime spectrum of L is continuous with respect to the lower Vietoris topology on $\mathrm{Sub}(A)$.

[1] P. Resende, J.P. Santos, *Linear structures on locales.* Theory Appl. Categ. 31 (2016) 502-541.

[2] P. Resende, J.P. Santos, *Open quotients of trivial vector bundles.* arXiv:1510.06329.

On limits and colimits of some subcategories of Loc

Lurdes Sousa

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The category of stably locally compact locales is precisely the Kan-injective hull, in **Loc**, of

$$F_0 \xrightarrow{f_0} F_1 \xleftarrow{f_2} F_2 \ ,$$

where F_k stands for the free frame generated by a set of cardinality k, and the f_k arrows are appropriate localic maps. If, for every cardinal n, we replace 2 by n, we obtain a chain of Kan-injective hulls, which moreover are KZ-monadic subcategories, and whose union is the entire category **Loc**. By means of the usual adjunction

$$\mathbf{Loc} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Top_0} \ ,$$

this chain gives rise to a chain of KZ-monadic subcategories of $\mathbf{Top_0}$ whose union is the category of sober spaces, and whose first element is the category of continuous lattices. Some of the Kleisli categories determined by these KZ-monadic subcategories are well known, namely, we have coherent locales for the stably locally compact ones, and algebraic lattices for continuous lattices. I will focus on the existence of (weighted) limits and colimits in the subcategories of the above chains and in their Kleisli categories.