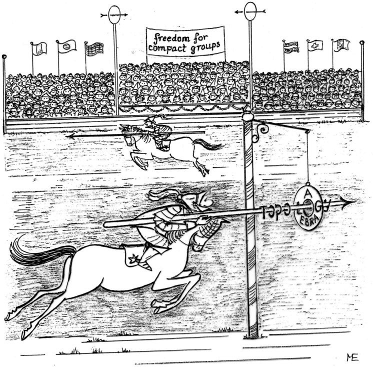
Workshop on Dualities 2016

University of Coimbra, Portugal
September 19–21, 2016



Dualities equivalent to the Ultrafilter Theorem

Marcel Erné

Faculty for Mathematics and Physics
Leibniz University Hannover
e-mail: erne@math.uni-hannover.de

September 23, 2016

Part I

Ultrafilters, compactness and sobriety

Ultrafilters, compactness and sobriety

The Banach Paradox

The Banach Paradox

The Banach Paradox

The Banach Paradox

The Banach Paradox

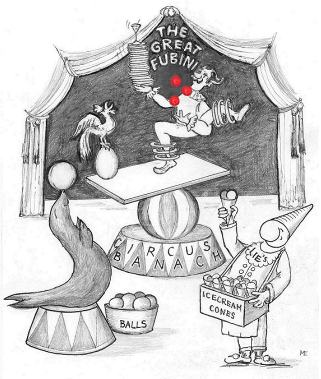
The Banach Paradox

The Banach Paradox

The Banach Paradox

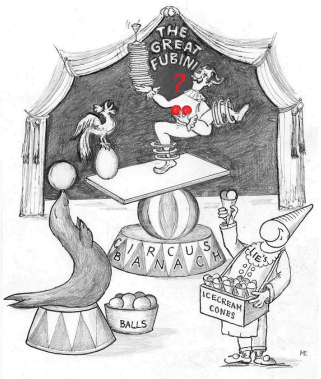
The Banach Paradox

The Banach Paradox



The Banach Paradox

The Banach Paradox



No Choice!


FREE CHOICE
SORRY, MY DEAR, WE HAVE NO CHOICE!
MOM, I WANT ONE OF THAT AND ONE OF THAT AND ONE OF THAT AND ONE OF THAT AND

No Choice!

If not otherwise stated, we do not assume the validity of any set-theoretical choice principles!

Weak Axioms of Choice

Two weakenings of the Axiom of Choice (**AC**) are

Weak Axioms of Choice

Two weakenings of the Axiom of Choice (**AC**) are

UP The Ultrafilter Principle: every proper set-theoretical filter is contained in an ultrafilter.

Weak Axioms of Choice

Two weakenings of the Axiom of Choice (**AC**) are

UP The Ultrafilter Principle: every proper set-theoretical filter is contained in an ultrafilter.

DC The Principle of Dependent Choices: if R is a relation on a set X such that for each $x \in X$ there is a $y \in X$ with $x \, R \, y$, then there is a sequence (x_n) in X with $x_n \, R \, x_{n+1}$ for all n.

Weak Axioms of Choice

Two weakenings of the Axiom of Choice (**AC**) are

UP The Ultrafilter Principle: every proper set-theoretical filter is contained in an ultrafilter.

DC The Principle of Dependent Choices: if R is a relation on a set X such that for each $x \in X$ there is a $y \in X$ with $x\,R\,y$, then there is a sequence (x_n) in X with $x_n\,R\,x_{n+1}$ for all n.

Theorem (Halpern and Levy 1964, Jech 1966, Pincus 1977)

UP *and* **DC** *are not only independent axioms in* **ZF** *or* **NBG** *set theory, but together they are still strictly weaker than* **AC**.

Lower sets, upper sets, cuts and feet

Let $P = (X, \leq)$ be a *quasiordered set* (qoset), with a reflexive and transitive relation $\leq$ on X. If $\leq$ is antisymmetric, P is a poset.

Lower sets, upper sets, cuts and feet

Let $P = (X, \leq)$ be a *quasiordered set* (qoset), with a reflexive and transitive relation $\leq$ on X. If $\leq$ is antisymmetric, P is a poset.

- The lower set resp. upper set generated by $Y \subseteq X$ is
  $\downarrow Y = \{x \in X : \exists y \in Y \ (x \leq y)\}$,
  $\uparrow Y = \{x \in X : \exists y \in Y \ (x \geq y)\}$.

Lower sets, upper sets, cuts and feet

Let $P = (X, \leq)$ be a *quasiordered set* (qoset), with a reflexive and transitive relation $\leq$ on X. If $\leq$ is antisymmetric, P is a poset.

- The lower set resp. upper set generated by $Y \subseteq X$ is
  $\downarrow Y = \{x \in X : \exists y \in Y \ (x \leq y)\}$,
  $\uparrow Y = \{x \in X : \exists y \in Y \ (x \geq y)\}$.
- The principal ideal resp. principal filter of $y \in X$ is
  $\downarrow y = \{x \in X : x \leq y\}$,
  $\uparrow y = \{x \in X : x \geq y\}$.

Lower sets, upper sets, cuts and feet

Let $P = (X, \leq)$ be a *quasiordered set* (qoset), with a reflexive and transitive relation $\leq$ on X. If $\leq$ is antisymmetric, P is a poset.

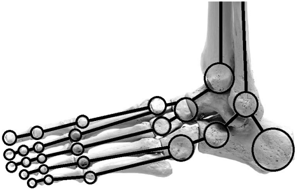
- The lower set resp. upper set generated by $Y \subseteq X$ is
  $\downarrow Y = \{x \in X : \exists y \in Y\ (x \leq y)\}$,
  $\uparrow Y = \{x \in X : \exists y \in Y\ (x \geq y)\}$.
- The principal ideal resp. principal filter of $y \in X$ is
  $\downarrow y = \{x \in X : x \leq y\}$,
  $\uparrow y = \{x \in X : x \geq y\}$.
- A foot is a finitely generated upper set (fus) $\uparrow F$.

Lower sets, upper sets, cuts and feet

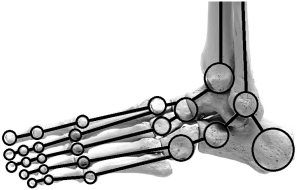
Let $P = (X, \leq)$ be a *quasiordered set* (qoset), with a reflexive and transitive relation $\leq$ on X. If $\leq$ is antisymmetric, P is a poset.

- The lower set resp. upper set generated by $Y \subseteq X$ is
  $\downarrow Y = \{x \in X : \exists y \in Y \ (x \leq y)\}$,
  $\uparrow Y = \{x \in X : \exists y \in Y \ (x \geq y)\}$.
- The principal ideal resp. principal filter of $y \in X$ is
  $\downarrow y = \{x \in X : x \leq y\}$,
  $\uparrow y = \{x \in X : x \geq y\}$.
- A foot is a finitely generated upper set (fus) $\uparrow F$.
- A poset is up-complete, a dcpo or a domain if all its directed subsets have joins.

Upper sets and (finite-bottomed) feet



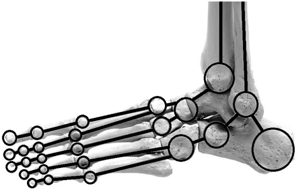
Upper sets and (finite-bottomed) feet



Lemma

(1) *The upper sets form the (upper) Alexandroff topology* αP.

Upper sets and (finite-bottomed) feet

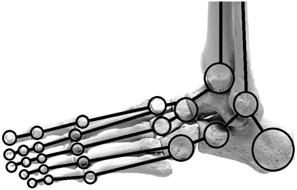


Lemma

(1) *The upper sets form the (upper) Alexandroff topology* αP.

(2) *The lower sets are exactly the closed sets w. r. t.* αP.

Upper sets and (finite-bottomed) feet



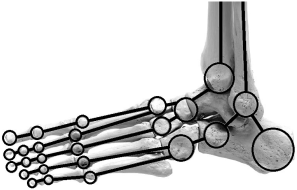
Lemma

(1) *The upper sets form the (upper) Alexandroff topology* αP.

(2) *The lower sets are exactly the closed sets w. r. t.* αP.

(3) *The feet are exactly the compact open sets w. r. t.* αP.

Upper sets and (finite-bottomed) feet



Lemma

(1) *The upper sets form the (upper) Alexandroff topology* αP.

(2) *The lower sets are exactly the closed sets w. r. t.* αP.

(3) *The feet are exactly the compact open sets w. r. t.* αP.

(4) *The feet, ordered by* $\supseteq$, *form the free semilattice* $F^{\uparrow}P$ *over* P.

The specialization order

Let X be a topological space.

The specialization order

Let X be a topological space.

- The specialization order of X is given by

$$x \leq y \;\Leftrightarrow\; x \in \overline{\{y\}} \;\Leftrightarrow\; \text{for all open } U\,(x \in U \Rightarrow y \in U).$$

The specialization order

Let X be a topological space.

- The specialization order of X is given by

$$x \leq y \;\Leftrightarrow\; x \in \overline{\{y\}} \;\Leftrightarrow\; \text{for all open } U\,(x \in U \Rightarrow y \in U).$$

Convention: *All order-theoretical statements about spaces refer to the specialization order, unless otherwise stated. In particular:*

The specialization order

Let X be a topological space.

- The specialization order of X is given by

$$x \leq y \;\Leftrightarrow\; x \in \overline{\{y\}} \;\Leftrightarrow\; \text{for all open } U\,(x \in U \Rightarrow y \in U).$$

Convention: *All order-theoretical statements about spaces refer to the specialization order, unless otherwise stated. In particular:*

- The saturation of a subset $Y \subseteq X$ is the intersection of all neighborhoods of Y; it is the upper set $\uparrow Y$ generated by Y.

The specialization order

Let X be a topological space.

- The specialization order of X is given by

$$x \leq y \;\Leftrightarrow\; x \in \overline{\{y\}} \;\Leftrightarrow\; \text{for all open } U\,(x \in U \Rightarrow y \in U).$$

Convention: *All order-theoretical statements about spaces refer to the specialization order, unless otherwise stated. In particular:*

- The saturation of a subset $Y \subseteq X$ is the intersection of all neighborhoods of Y; it is the upper set $\uparrow Y$ generated by Y.
- The (neighborhood) core of a point $x \in X$ is the principal filter $\uparrow x$, the intersection of all neighborhoods of x.

The specialization order

Let X be a topological space.

- The specialization order of X is given by

$$x \leq y \;\Leftrightarrow\; x \in \overline{\{y\}} \;\Leftrightarrow\; \text{ for all open } U\,(x \in U \Rightarrow y \in U).$$

Convention: *All order-theoretical statements about spaces refer to the specialization order, unless otherwise stated. In particular:*

- The saturation of a subset $Y \subseteq X$ is the intersection of all neighborhoods of Y; it is the upper set $\uparrow Y$ generated by Y.
- The (neighborhood) core of a point $x \in X$ is the principal filter $\uparrow x$, the intersection of all neighborhoods of x.
- Dually, the point closure of x is the principal ideal $\downarrow x$.

Monotonicity properties

Let X be a topological space.

Monotonicity properties

Let X be a topological space.

- A monotone net in X is a map ν from a directed set N into X such that $m \leq n$ implies $\nu(m) \leq \nu(n)$ (w.r.t. specialization).

Monotonicity properties

Let X be a topological space.

- A monotone net in X is a map ν from a directed set N into X such that $m \leq n$ implies $\nu(m) \leq \nu(n)$ (w.r.t. specialization).
- X is a monotone convergence space if it is T_0 and every monotone net in X has a join to which it converges.

Monotonicity properties

Let X be a topological space.

- A monotone net in X is a map ν from a directed set N into X such that $m \leq n$ implies $\nu(m) \leq \nu(n)$ (w.r.t. specialization).
- X is a monotone convergence space if it is T_0 and every monotone net in X has a join to which it converges.
- X is a d-space if the closure of any directed subset is the closure of a unique point.

Monotonicity properties

Let X be a topological space.

- A monotone net in X is a map ν from a directed set N into X such that $m \leq n$ implies $\nu(m) \leq \nu(n)$ (w.r.t. specialization).
- X is a monotone convergence space if it is T_0 and every monotone net in X has a join to which it converges.
- X is a d-space if the closure of any directed subset is the closure of a unique point.
- X is monotone determined if a subset U is open whenever any monotone net converging to a point of U is eventually in U.

A Min-Max characterization of Scott spaces

A Min-Max characterization of Scott spaces

- A subset U of a poset $P = (X, \leq)$ is Scott open if for all directed sets $D \subseteq P$ possessing a join, $D \cap U \neq \emptyset \Leftrightarrow \bigvee D \in U$.

A Min-Max characterization of Scott spaces

- A subset U of a poset $P = (X, \leq)$ is Scott open if for all directed sets $D \subseteq P$ possessing a join, $D \cap U \neq \emptyset \Leftrightarrow \bigvee D \in U$.
- The Scott topology σP consists of all Scott-open sets.

A Min-Max characterization of Scott spaces

- A subset U of a poset $P = (X, \leq)$ is Scott open if for all directed sets $D \subseteq P$ possessing a join, $D \cap U \neq \emptyset \Leftrightarrow \bigvee D \in U$.
- The Scott topology σP consists of all Scott-open sets.
- The Scott space is $\Sigma P = (X, \sigma P)$.

A Min-Max characterization of Scott spaces

- A subset U of a poset $P = (X, \leq)$ is Scott open if for all directed sets $D \subseteq P$ possessing a join, $D \cap U \neq \emptyset \Leftrightarrow \bigvee D \in U$.
- The Scott topology σP consists of all Scott-open sets.
- The Scott space is $\Sigma P = (X, \sigma P)$.

Theorem (ME 2012)

On a domain, the Scott topology is the finest topology making it a monotone convergence space and the corsest topology making it a monotone determined space. Hence, the Scott spaces of domains are exactly the monotone determined monotone convergence spaces.

Ideals, filters and prime elements

- An ideal of a qoset is an (up-)directed lower set.

Ideals, filters and prime elements

- An ideal of a qoset is an (up-)directed lower set.
- A filter of a qoset is a filtered (i.e. down-directed) upper set.

Ideals, filters and prime elements

- An ideal of a qoset is an (up-)directed lower set.
- A filter of a qoset is a filtered (i.e. down-directed) upper set.
- An element p of a qoset is prime if the complement of the principal ideal $\downarrow p$ is a filter.
- A locale or frame is a complete lattice L satisfying the infinite distributive law $a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$.

Ideals, filters and prime elements

- An ideal of a qoset is an (up-)directed lower set.
- A filter of a qoset is a filtered (i.e. down-directed) upper set.
- An element p of a qoset is prime if the complement of the principal ideal $\downarrow p$ is a filter.
- A locale or frame is a complete lattice L satisfying the infinite distributive law $a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$.
- A complete lattice is spatial if each of its elements is a meet of primes.

Ideals, filters and prime elements

- An ideal of a qoset is an (up-)directed lower set.
- A filter of a qoset is a filtered (i.e. down-directed) upper set.
- An element p of a qoset is prime if the complement of the principal ideal $\downarrow p$ is a filter.
- A locale or frame is a complete lattice L satisfying the infinite distributive law $a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$.
- A complete lattice is spatial if each of its elements is a meet of primes.

Lemma

(1) *Every spatial lattice is a frame, but not conversely.*

Ideals, filters and prime elements

- An ideal of a qoset is an (up-)directed lower set.
- A filter of a qoset is a filtered (i.e. down-directed) upper set.
- An element p of a qoset is prime if the complement of the principal ideal $\downarrow p$ is a filter.
- A locale or frame is a complete lattice L satisfying the infinite distributive law $a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$.
- A complete lattice is spatial if each of its elements is a meet of primes.

Lemma

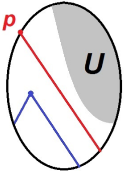
(1) *Every spatial lattice is a frame, but not conversely.*
(2) *The spatial lattices are, up to isomorphism, just the topologies.*

The Separation Lemma for Locales

- A complete lattice L enjoys the Strong Prime Element Theorem or the Separation Lemma (**SL**) if each element outside a Scott-open filter U in L is below a prime element outside U.

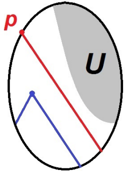
The Separation Lemma for Locales

- A complete lattice L enjoys the Strong Prime Element Theorem or the Separation Lemma (**SL**) if each element outside a Scott-open filter U in L is below a prime element outside U.



The Separation Lemma for Locales

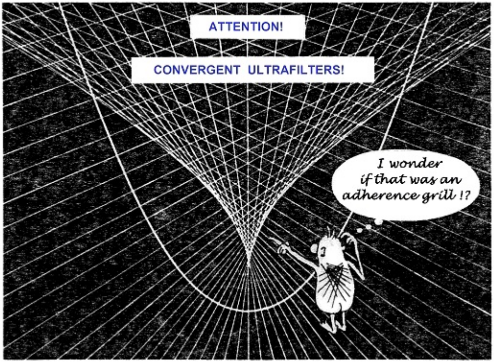
- A complete lattice L enjoys the Strong Prime Element Theorem or the Separation Lemma (**SL**) if each element outside a Scott-open filter U in L is below a prime element outside U.



Theorem (ME 1986, BB & ME 1993)

The Ultrafilter Principle (**UP**) *and the Prime Ideal Theorem* (**PIT**) *are equivalent to the Separation Lemma for Locales (or Quantales).*

The ultrafilter paradox



We shall not talk about ultrafilters,
but only about the Separation Lemma!

Hyper- and supercompactness

- A subset C of a space X is hypercompact if $\uparrow C$ is a foot $\uparrow F$.

Hyper- and supercompactness

- A subset C of a space X is hypercompact if $\uparrow C$ is a foot $\uparrow F$.
- A subset C of a space X is supercompact if $\uparrow C$ is a core $\uparrow x$.

Hyper- and supercompactness

- A subset C of a space X is hypercompact if $\uparrow C$ is a foot $\uparrow F$.
- A subset C of a space X is supercompact if $\uparrow C$ is a core $\uparrow x$.
- A space is compactly based resp. hypercompactly based resp. supercompactly based if it has a base of compact resp. hypercompact resp. supercompact open sets.

Hyper- and supercompactness

- A subset C of a space X is hypercompact if $\uparrow C$ is a foot $\uparrow F$.
- A subset C of a space X is supercompact if $\uparrow C$ is a core $\uparrow x$.
- A space is compactly based resp. hypercompactly based resp. supercompactly based if it has a base of compact resp. hypercompact resp. supercompact open sets.
- A space is locally compact resp. locally hypercompact resp. locally supercompact if each point has a neighborhood base of compact resp. hypercompact resp. supercompact sets.

Sober spaces

- A topological space is sober if each irreducible closed set is the closure of a unique point.

Sober spaces

- A topological space is sober if each irreducible closed set is the closure of a unique point.

Lemma

(1) *The category of sober spaces is dual to that of spatial frames.*

Sober spaces

- A topological space is sober if each irreducible closed set is the closure of a unique point.

Lemma

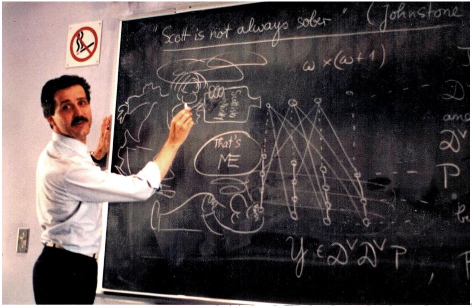
(1) *The category of sober spaces is dual to that of spatial frames.*
(2) *Scott spaces of domains are d-spaces but not always sober.*
(Johnstone 1980)

Sober spaces

- A topological space is sober if each irreducible closed set is the closure of a unique point.

Lemma

(1) *The category of sober spaces is dual to that of spatial frames.*
(2) *Scott spaces of domains are d-spaces but not always sober.*
(Johnstone 1980)



The power of sobriety

A sober space and
a non-sober neighborhood

Sobrification

δ-sobriety and well-filtration

- The Scott-open filters of a poset P form the Lawson dual δP.

δ-sobriety and well-filtration

- The Scott-open filters of a poset P form the Lawson dual δP.
- A T_0-space X is
  - δ-sober if each Scott-open filter of open sets (that is, each $\mathcal{V} \in \delta\,\mathcal{O}X$) contains all open neighborhoods of its intersection.

δ-sobriety and well-filtration

- The Scott-open filters of a poset P form the Lawson dual δP.
- A T_0-space X is
  - δ-sober if each Scott-open filter of open sets (that is, each $\mathcal{V} \in \delta\,\mathcal{O}X$) contains all open neighborhoods of its intersection.
  - well-filtered (resp. $\mathcal{H}$-well-filtered) if for any filter base $\mathcal{B}$ of compact (resp. hypercompact) saturated sets, each open neighborhood of the intersection $\bigcap \mathcal{B}$ contains a member of $\mathcal{B}$.

δ-sobriety and well-filtration

- The Scott-open filters of a poset P form the Lawson dual δP.
- A T_0-space X is
  - δ-sober if each Scott-open filter of open sets (that is, each $\mathcal{V} \in \delta\,\mathcal{O}X$) contains all open neighborhoods of its intersection.
  - well-filtered (resp. $\mathcal{H}$-well-filtered) if for any filter base $\mathcal{B}$ of compact (resp. hypercompact) saturated sets, each open neighborhood of the intersection $\bigcap \mathcal{B}$ contains a member of $\mathcal{B}$.

Theorem (ME 1979/2005, Hofmann & Mislove 1981)

(1) *Every δ-sober space is sober and well-filtered.*

δ-sobriety and well-filtration

- The Scott-open filters of a poset P form the Lawson dual δP.
- A T_0-space X is
  - δ-sober if each Scott-open filter of open sets (that is, each $\mathcal{V} \in \delta\,\mathcal{O}X$) contains all open neighborhoods of its intersection.
  - well-filtered (resp. $\mathcal{H}$-well-filtered) if for any filter base $\mathcal{B}$ of compact (resp. hypercompact) saturated sets, each open neighborhood of the intersection $\bigcap \mathcal{B}$ contains a member of $\mathcal{B}$.

Theorem (ME 1979/2005, Hofmann & Mislove 1981)

(1) *Every δ-sober space is sober and well-filtered.*

(2) *Every locally compact well-filtered space is δ-sober.*

δ-sobriety and well-filtration

- The Scott-open filters of a poset P form the Lawson dual δP.
- A T_0-space X is
  - δ-sober if each Scott-open filter of open sets (that is, each $\mathcal{V} \in \delta\,\mathcal{O}X$) contains all open neighborhoods of its intersection.
  - well-filtered (resp. $\mathcal{H}$-well-filtered) if for any filter base $\mathcal{B}$ of compact (resp. hypercompact) saturated sets, each open neighborhood of the intersection $\bigcap \mathcal{B}$ contains a member of $\mathcal{B}$.

Theorem (ME 1979/2005, Hofmann & Mislove 1981)

(1) *Every δ-sober space is sober and well-filtered.*

(2) *Every locally compact well-filtered space is δ-sober.*

(3) *X is δ-sober iff the locale $\mathcal{O}X$ enjoys the Separation Lemma.*

The power of δ-sobriety and Tychonoff's Theorem

Theorem (ME 2012)

Each of the following statements is equivalent to **UP***:*

The power of δ-sobriety and Tychonoff's Theorem

Theorem (ME 2012)

Each of the following statements is equivalent to **UP**:

- *The Hofmann-Mislove Theorem: sober spaces are δ-sober.*

The power of δ-sobriety and Tychonoff's Theorem

Theorem (ME 2012)

Each of the following statements is equivalent to **UP***:*

- *The Hofmann-Mislove Theorem: sober spaces are δ-sober.*
- *Sober spaces are well-filtered.*

The power of δ-sobriety and Tychonoff's Theorem

Theorem (ME 2012)

Each of the following statements is equivalent to **UP***:*

- *The Hofmann-Mislove Theorem: sober spaces are δ-sober.*
- *Sober spaces are well-filtered.*
- *Every filter base of compact saturated sets in a sober space has a nonempty intersection.*

The power of δ-sobriety and Tychonoff's Theorem

Theorem (ME 2012)

Each of the following statements is equivalent to **UP***:*

- *The Hofmann-Mislove Theorem: sober spaces are δ-sober.*
- *Sober spaces are well-filtered.*
- *Every filter base of compact saturated sets in a sober space has a nonempty intersection.*
- *Every filter base of compact saturated sets in a sober space has a compact intersection.*

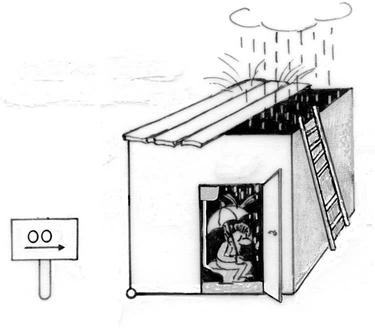
The power of δ-sobriety and Tychonoff's Theorem

Theorem (ME 2012)

Each of the following statements is equivalent to **UP***:*

- *The Hofmann-Mislove Theorem: sober spaces are δ-sober.*
- *Sober spaces are well-filtered.*
- *Every filter base of compact saturated sets in a sober space has a nonempty intersection.*
- *Every filter base of compact saturated sets in a sober space has a compact intersection.*
- *Tychonoff's Product Theorem for any class of sober spaces containing a two-element discrete space.*

Tychonoff's Product Theorem



Compact unit cube
with a partial
open covering

The power of Rudin's Lemma

- For any system $\mathcal{Y}$ of sets, each member of the system $\mathcal{Y}^{\#} = \{Z \subseteq \bigcup \mathcal{Y} : \forall\, Y \in \mathcal{Y}\,(Y \cap Z \neq \emptyset)\}$ is a transversal of $\mathcal{Y}$.

The power of Rudin's Lemma

- For any system $\mathcal{Y}$ of sets, each member of the system $\mathcal{Y}^{\#} = \{Z \subseteq \bigcup \mathcal{Y} : \forall\, Y \in \mathcal{Y}\,(Y \cap Z \neq \emptyset)\}$ is a transversal of $\mathcal{Y}$.

Theorem (ME 2012)

Each of the following statements is equivalent to **UP***:*

The power of Rudin's Lemma

- For any system $\mathcal{Y}$ of sets, each member of the system $\mathcal{Y}^{\#} = \{Z \subseteq \bigcup \mathcal{Y} : \forall\, Y \in \mathcal{Y}\,(Y \cap Z \neq \emptyset)\}$ is a transversal of $\mathcal{Y}$.

Theorem (ME 2012)

Each of the following statements is equivalent to **UP***:*

- *Every system of compact sets whose saturations form a filter base has an irreducible transversal.*

The power of Rudin's Lemma

- For any system $\mathcal{Y}$ of sets, each member of the system $\mathcal{Y}^{\#} = \{Z \subseteq \bigcup \mathcal{Y} : \forall\, Y \in \mathcal{Y}\, (Y \cap Z \neq \emptyset)\}$ is a transversal of $\mathcal{Y}$.

Theorem (ME 2012)

Each of the following statements is equivalent to **UP***:*

- *Every system of compact sets whose saturations form a filter base has an irreducible transversal.*
- *Rudin's Lemma: Every system of finite sets generating a filter base of feet has a directed transversal.*

The power of Rudin's Lemma

- For any system $\mathcal{Y}$ of sets, each member of the system $\mathcal{Y}^{\#} = \{Z \subseteq \bigcup \mathcal{Y} : \forall\, Y \in \mathcal{Y}\,(Y \cap Z \neq \emptyset)\}$ is a transversal of $\mathcal{Y}$.

Theorem (ME 2012)

Each of the following statements is equivalent to **UP***:*

- *Every system of compact sets whose saturations form a filter base has an irreducible transversal.*
- *Rudin's Lemma: Every system of finite sets generating a filter base of feet has a directed transversal.*
- *Every d-space is $\mathcal{H}$-well-filtered.*

The power of Rudin's Lemma

- For any system $\mathcal{Y}$ of sets, each member of the system $\mathcal{Y}^{\#} = \{Z \subseteq \bigcup \mathcal{Y} : \forall\, Y \in \mathcal{Y}\,(Y \cap Z \neq \emptyset)\}$ is a transversal of $\mathcal{Y}$.

Theorem (ME 2012)

Each of the following statements is equivalent to **UP***:*

- *Every system of compact sets whose saturations form a filter base has an irreducible transversal.*
- *Rudin's Lemma: Every system of finite sets generating a filter base of feet has a directed transversal.*
- *Every d-space is $\mathcal{H}$-well-filtered.*
- *Every locally hypercompact d-space space is well-filtered.*

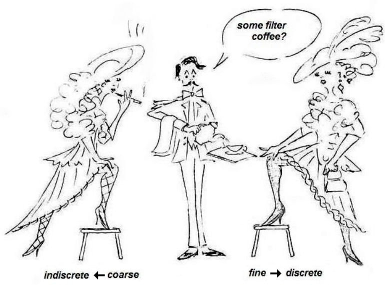
The power of Rudin's Lemma

A transversal for a filter base of feet

Part II

Order-topological dualities

Nets and filters



Coffee break?

Order-topological dualities

Compactness properties in posets

Let P be a poset and c an element of P.

Compactness properties in posets

Let P be a poset and c an element of P.

- c is compact if $P \setminus \uparrow c$ is Scott closed.

Compactness properties in posets

Let P be a poset and c an element of P.

- c is compact if $P \setminus \uparrow c$ is Scott closed.
- c is hypercompact if $P \setminus \uparrow c$ is a finitely generated lower set.

Compactness properties in posets

Let P be a poset and c an element of P.

- c is compact if $P \setminus \uparrow c$ is Scott closed.
- c is hypercompact if $P \setminus \uparrow c$ is a finitely generated lower set.
- c is supercompact if $P \setminus \uparrow c$ is a principal ideal.

Compactness properties in posets

Let P be a poset and c an element of P.

- c is compact if $P \setminus \uparrow c$ is Scott closed.
- c is hypercompact if $P \setminus \uparrow c$ is a finitely generated lower set.
- c is supercompact if $P \setminus \uparrow c$ is a principal ideal.

Lemma

These definitions generalize the corresponding topological ones.

Compactness properties in posets

Let P be a poset and c an element of P.

- c is compact if $P \setminus \uparrow c$ is Scott closed.
- c is hypercompact if $P \setminus \uparrow c$ is a finitely generated lower set.
- c is supercompact if $P \setminus \uparrow c$ is a principal ideal.

Lemma

These definitions generalize the corresponding topological ones.

- P is quasialgebraic if each principal filter is the intersection of a filterbase of Scott-open feet.

Compactness properties in posets

Let P be a poset and c an element of P.

- c is compact if $P \setminus \uparrow c$ is Scott closed.
- c is hypercompact if $P \setminus \uparrow c$ is a finitely generated lower set.
- c is supercompact if $P \setminus \uparrow c$ is a principal ideal.

Lemma

These definitions generalize the corresponding topological ones.

- P is quasialgebraic if each principal filter is the intersection of a filterbase of Scott-open feet.
- P is algebraic resp. hyperalgebraic if each of its elements is a directed join of compact resp. hypercompact elements.

Compactness properties in posets

Let P be a poset and c an element of P.

- c is compact if $P \setminus {\uparrow} c$ is Scott closed.
- c is hypercompact if $P \setminus {\uparrow} c$ is a finitely generated lower set.
- c is supercompact if $P \setminus {\uparrow} c$ is a principal ideal.

Lemma

These definitions generalize the corresponding topological ones.

- P is quasialgebraic if each principal filter is the intersection of a filterbase of Scott-open feet.
- P is algebraic resp. hyperalgebraic if each of its elements is a directed join of compact resp. hypercompact elements.
- P is superalgebraic if it is complete and each of its elements is a join of supercompact elements.

Continuity properties of domains

A domain P is

Continuity properties of domains

A domain P is

- quasicontinuous if each principal filter $\uparrow x$ is the intersection of a filterbase of feet having x in their Scott-interior.

Continuity properties of domains

A domain P is

- quasicontinuous if each principal filter $\uparrow x$ is the intersection of a filterbase of feet having x in their Scott-interior.
- continuous if each principal filter $\uparrow x$ is the intersection of a filterbase of cores having x in their Scott-interior.

Continuity properties of domains

A domain P is

- quasicontinuous if each principal filter $\uparrow x$ is the intersection of a filterbase of feet having x in their Scott-interior.
- continuous if each principal filter $\uparrow x$ is the intersection of a filterbase of cores having x in their Scott-interior.
- hypercontinuous if each element x is the directed join of $\{y \in P : \exists\, F \subset_\omega P\ (x \in L \setminus {\downarrow} F \subseteq {\uparrow} y)\}$.

Continuity properties of domains

A domain P is

- quasicontinuous if each principal filter $\uparrow x$ is the intersection of a filterbase of feet having x in their Scott-interior.
- continuous if each principal filter $\uparrow x$ is the intersection of a filterbase of cores having x in their Scott-interior.
- hypercontinuous if each element x is the directed join of $\{y \in P : \exists\, F \subset_\omega P\ (x \in L \setminus {\downarrow} F \subseteq {\uparrow} y)\}$.
- supercontinuous if it is complete and each $x \in P$ is the join of $\{y \in P : \exists\, z \in P\ (x \in P \setminus {\downarrow} z \subseteq {\uparrow} y)\}$.

δ-continuous domains

To avoid choice, we consider a variant of continuous domains:

- A poset P is called δ-continuous if each of its elements y is the directed join of all elements x for which there exists a $U \in \delta P$ with $y \in U \subseteq \uparrow x$.

δ-continuous domains

To avoid choice, we consider a variant of continuous domains:

- A poset P is called δ-continuous if each of its elements y is the directed join of all elements x for which there exists a $U \in \delta P$ with $y \in U \subseteq \uparrow x$.

Lemma

Every δ-continuous domain is continuous. **DC** *implies the converse.*

δ-continuous domains

To avoid choice, we consider a variant of continuous domains:

- A poset P is called δ-continuous if each of its elements y is the directed join of all elements x for which there exists a $U \in \delta P$ with $y \in U \subseteq \uparrow x$.

Lemma

Every δ-continuous domain is continuous. **DC** *implies the converse.*

Theorem (ME 2012; in ZFC, no δ: Lawson 1979)

The category of δ-continuous domains and δ-continuous maps is self-dual under the functor δ.

δ-continuous domains

To avoid choice, we consider a variant of continuous domains:

- A poset P is called δ-continuous if each of its elements y is the directed join of all elements x for which there exists a $U \in \delta P$ with $y \in U \subseteq \uparrow x$.

Lemma

Every δ-continuous domain is continuous. **DC** *implies the converse.*

Theorem (ME 2012; in ZFC, no δ: Lawson 1979)

The category of δ-continuous domains and δ-continuous maps is self-dual under the functor δ.

Remark (ME 2009) *In a suitable terminology, the δ-continuous domains are the sober enveloped locally compact preframes.*

Some elementary order-theoretical facts

Lemma

(1) *For domains, one has the following implications:*

$$\begin{array}{ccccccc}
\textit{superalgebraic} & \Rightarrow & \textit{hyperalgebraic} & \Rightarrow & \textit{algebraic} & \Rightarrow & \textit{quasialgebraic} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\textit{supercontinuous} & \Rightarrow & \textit{hypercontinuous} & \Rightarrow & \textit{continuous} & \Rightarrow & \textit{quasicontinuous.}
\end{array}$$

Some elementary order-theoretical facts

Lemma

(1) *For domains, one has the following implications:*

$$
\begin{array}{ccccccc}
\textit{superalgebraic} & \Rightarrow & \textit{hyperalgebraic} & \Rightarrow & \textit{algebraic} & \Rightarrow & \textit{quasialgebraic} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\textit{supercontinuous} & \Rightarrow & \textit{hypercontinuous} & \Rightarrow & \textit{continuous} & \Rightarrow & \textit{quasicontinuous.}
\end{array}
$$

(2) *Every completely distributive complete lattice, satisfying*

$$\bigwedge \{\bigvee Y_i : i \in I\} = \bigvee \{\bigwedge_{i\in I} x_i : x \in \prod_{i\in I} Y_i\}$$

for arbitrary families of subsets Y_i, is supercontinuous.

The converse is equivalent to **AC**.

Some elementary order-theoretical facts

Lemma

(1) *For domains, one has the following implications:*

$$\begin{array}{ccccccc}
\textit{superalgebraic} & \Rightarrow & \textit{hyperalgebraic} & \Rightarrow & \textit{algebraic} & \Rightarrow & \textit{quasialgebraic} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\textit{supercontinuous} & \Rightarrow & \textit{hypercontinuous} & \Rightarrow & \textit{continuous} & \Rightarrow & \textit{quasicontinuous.}
\end{array}$$

(2) *Every completely distributive complete lattice, satisfying*

$$\bigwedge \{\bigvee Y_i : i \in I\} = \bigvee \{\bigwedge_{i\in I} x_i : x \in \prod_{i\in I} Y_i\}$$

for arbitrary families of subsets Y_i, is supercontinuous.

The converse is equivalent to **AC**.

(3) **DC** *implies that every supercontinuous lattice is spatial.*

Some elementary order-theoretical facts

Lemma

(1) *For domains, one has the following implications:*

$$\begin{array}{ccccccc}
\textit{superalgebraic} & \Rightarrow & \textit{hyperalgebraic} & \Rightarrow & \textit{algebraic} & \Rightarrow & \textit{quasialgebraic} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\textit{supercontinuous} & \Rightarrow & \textit{hypercontinuous} & \Rightarrow & \textit{continuous} & \Rightarrow & \textit{quasicontinuous.}
\end{array}$$

(2) *Every completely distributive complete lattice, satisfying*

$$\bigwedge \{\bigvee Y_i : i \in I\} = \bigvee \{\bigwedge_{i \in I} x_i : x \in \prod_{i \in I} Y_i\}$$

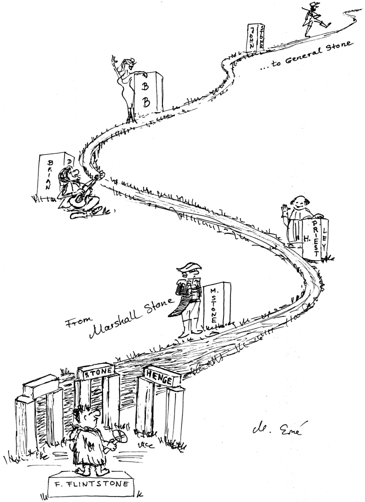
for arbitrary families of subsets Y_i, is supercontinuous.

The converse is equivalent to **AC**.

(3) **DC** *implies that every supercontinuous lattice is spatial.*

(4) *Algebraic posets are δ-continuous.*

Mile-Stones of duality



Continuous domains and locally compact spaces

Theorem (ME 2015)

Via the open set functor and the spectrum functor,

Continuous domains and locally compact spaces

Theorem (ME 2015)

Via the open set functor and the spectrum functor,

(1) *the category of δ-continuous frames enjoying* **SL** *is dual to the category of δ-sober locally compact spaces,*

Continuous domains and locally compact spaces

Theorem (ME 2015)

Via the open set functor and the spectrum functor,

(1) *the category of δ-continuous frames enjoying* **SL** *is dual to the category of δ-sober locally compact spaces,*

(2) *the category of algebraic frames enjoying* **SL** *is equivalent to the category of prime ideal separated semilattices and dual to the category of δ-sober compactly based spaces,*

Continuous domains and locally compact spaces

Theorem (ME 2015)

Via the open set functor and the spectrum functor,

1. *the category of δ-continuous frames enjoying* **SL** *is dual to the category of δ-sober locally compact spaces,*
2. *the category of algebraic frames enjoying* **SL** *is equivalent to the category of prime ideal separated semilattices and dual to the category of δ-sober compactly based spaces,*
3. *the category of coherent frames enjoying* **SL** *is equivalent to the category of prime ideal separated bounded lattices and dual to the category of δ-sober spectral spaces.*

Continuous domains and locally compact spaces

Theorem (ME 2015)

Via the open set functor and the spectrum functor,

1. *the category of δ-continuous frames enjoying* **SL** *is dual to the category of δ-sober locally compact spaces,*
2. *the category of algebraic frames enjoying* **SL** *is equivalent to the category of prime ideal separated semilattices and dual to the category of δ-sober compactly based spaces,*
3. *the category of coherent frames enjoying* **SL** *is equivalent to the category of prime ideal separated bounded lattices and dual to the category of δ-sober spectral spaces.*

Replacing 'δ-sober' with 'sober' and 'prime ideal separated' with 'distributive' makes these facts equivalent to **UP***.*

In **ZFC**, without δ and **SL**, (1) is due to Hofmann & Lawson (1978).

Topological characterization of domains

Theorem (ME 1991/2012)

(1) *In* **ZF**, *the category of algebraic domains is isomorphic to the category of supercompactly based sober spaces and dual to the category of superalgebraic frames.*

Topological characterization of domains

Theorem (ME 1991/2012)

(1) *In* **ZF**, *the category of algebraic domains is isomorphic to the category of supercompactly based sober spaces and dual to the category of superalgebraic frames.*

(2) *In* **ZF**, *the category of continuous domains is isomorphic to the category of locally supercompact sober spaces and dual to the category of supercontinuous spatial frames.*

Topological characterization of domains

Theorem (ME 1991/2012)

(1) *In* **ZF**, *the category of algebraic domains is isomorphic to the category of supercompactly based sober spaces and dual to the category of superalgebraic frames.*

(2) *In* **ZF**, *the category of continuous domains is isomorphic to the category of locally supercompact sober spaces and dual to the category of supercontinuous spatial frames.*

(3) **UP** $\Rightarrow$ *the category of quasialgebraic domains is isomorphic to the category of hypercompactly based sober spaces and dual to the category of hyperalgebraic frames.*

Topological characterization of domains

Theorem (ME 1991/2012)

(1) *In* **ZF**, *the category of algebraic domains is isomorphic to the category of supercompactly based sober spaces and dual to the category of superalgebraic frames.*

(2) *In* **ZF**, *the category of continuous domains is isomorphic to the category of locally supercompact sober spaces and dual to the category of supercontinuous spatial frames.*

(3) **UP** $\Rightarrow$ *the category of quasialgebraic domains is isomorphic to the category of hypercompactly based sober spaces and dual to the category of hyperalgebraic frames.*

(4) **UP** $\Rightarrow$ *the category of quasicontinuous domains is isomorphic to the category of locally hypercompact sober spaces and dual to the category of hypercontinuous frames.*

Topological characterization of domains

Theorem (ME 1991/2012)

(1) **UP** $\Leftrightarrow$ *the category of algebraic domains is isomorphic to the category of supercompactly based* δ*-sober spaces and dual to the category of superalgebraic frames.*

Topological characterization of domains

Theorem (ME 1991/2012)

(1) **UP** $\Leftrightarrow$ *the category of algebraic domains is isomorphic to the category of supercompactly based δ-sober spaces and dual to the category of superalgebraic frames.*

(2) **UP** $\Leftrightarrow$ *the category of continuous domains is isomorphic to the category of locally supercompact δ-sober spaces and dual to the category of supercontinuous spatial frames.*

Topological characterization of domains

Theorem (ME 1991/2012)

(1) **UP** $\Leftrightarrow$ *the category of algebraic domains is isomorphic to the category of supercompactly based* δ*-sober spaces and dual to the category of superalgebraic frames.*

(2) **UP** $\Leftrightarrow$ *the category of continuous domains is isomorphic to the category of locally supercompact* δ*-sober spaces and dual to the category of supercontinuous spatial frames.*

(3) **UP** $\Leftrightarrow$ *the category of quasialgebraic domains is isomorphic to the category of hypercompactly based* δ*-sober spaces and dual to the category of hyperalgebraic frames.*

Topological characterization of domains

Theorem (ME 1991/2012)

(1) **UP** $\Leftrightarrow$ *the category of algebraic domains is isomorphic to the category of supercompactly based* δ*-sober spaces and dual to the category of superalgebraic frames.*

(2) **UP** $\Leftrightarrow$ *the category of continuous domains is isomorphic to the category of locally supercompact* δ*-sober spaces and dual to the category of supercontinuous spatial frames.*

(3) **UP** $\Leftrightarrow$ *the category of quasialgebraic domains is isomorphic to the category of hypercompactly based* δ*-sober spaces and dual to the category of hyperalgebraic frames.*

(4) **UP** $\Leftrightarrow$ *the category of quasicontinuous domains is isomorphic to the category of locally hypercompact* δ*-sober spaces and dual to the category of hypercontinuous frames.*

Characterization of hypercompactly based spaces

Theorem (ME 2009)

The following hold in **ZFC** *(set theory with the Axiom of Choice):*

Characterization of hypercompactly based spaces

Theorem (ME 2009)

The following hold in **ZFC** *(set theory with the Axiom of Choice):*

(1) *Compact open subsets of monotone determined spaces are hypercompact.*

Characterization of hypercompactly based spaces

Theorem (ME 2009)

The following hold in **ZFC** *(set theory with the Axiom of Choice):*

(1) *Compact open subsets of monotone determined spaces are hypercompact.*

(2) *The hypercompactly based spaces are exactly the compactly based monotone determined spaces.*

Characterization of hypercompactly based spaces

Theorem (ME 2009)

The following hold in **ZFC** *(set theory with the Axiom of Choice):*

(1) *Compact open subsets of monotone determined spaces are hypercompact.*

(2) *The hypercompactly based spaces are exactly the compactly based monotone determined spaces.*

(3) *The hypercompactly based sober spaces are exactly the compactly based Scott spaces of domains.*

Characterization of hypercompactly based spaces

Theorem (ME 2009)

The following hold in **ZFC** *(set theory with the Axiom of Choice):*

(1) *Compact open subsets of monotone determined spaces are hypercompact.*

(2) *The hypercompactly based spaces are exactly the compactly based monotone determined spaces.*

(3) *The hypercompactly based sober spaces are exactly the compactly based Scott spaces of domains.*

Coherence for spaces

A topological space is

- coherent if finite intersections of compact upper sets are compact.

Coherence for spaces

A topological space is

- coherent if finite intersections of compact upper sets are compact.

Coherence for spaces

A topological space is

- coherent if finite intersections of compact upper sets are compact.
- open coherent if finite intersections of compact open sets are compact.

Coherence for spaces

A topological space is

- coherent if finite intersections of compact upper sets are compact.
- open coherent if finite intersections of compact open sets are compact.
- A spectral (hyperspectral, superspectral) space is a compactly (hypercompactly, supercompactly) based coherent sober space.

Lemma

For Stone spaces, i.e. compactly based sober spaces, the above two notions of coherence are equivalent.

Coherence for domains and lattices

- A poset P is quasicoherent if it is quasialgebraic and the Scott space ΣP is coherent.
- A complete lattice is coherent (hypercoherent, supercoherent) if it is algebraic (hyperalgebraic, superalgebraic) and finite meets of compact elements are compact.

Lemma

Not only every coherent, but even every algebraic complete lattice is quasicoherent.

Finite prime decompositions

Let S be a meet-semilattice.

Finite prime decompositions

Let S be a meet-semilattice.

- A prime ideal of S is a directed proper lower set whose complement is a subsemilattice.

Finite prime decompositions

Let S be a meet-semilattice.

- A prime ideal of S is a directed proper lower set whose complement is a subsemilattice.
- S is an np semilattice if its prime ideal poset is noetherian, i.e. any directed set of prime ideals has a greatest member.

Finite prime decompositions

Let S be a meet-semilattice.

- A prime ideal of S is a directed proper lower set whose complement is a subsemilattice.
- S is an np semilattice if its prime ideal poset is noetherian, i.e. any directed set of prime ideals has a greatest member.
- S is an fp semilattice if each element is a finite meet of primes.

Finite prime decompositions

Let S be a meet-semilattice.

- A prime ideal of S is a directed proper lower set whose complement is a subsemilattice.
- S is an np semilattice if its prime ideal poset is noetherian, i.e. any directed set of prime ideals has a greatest member.
- S is an fp semilattice if each element is a finite meet of primes.
- S is an fpi semilattice if each principal ideal is a finite meet of prime ideals.

Finite prime decompositions

Let S be a meet-semilattice.

- A prime ideal of S is a directed proper lower set whose complement is a subsemilattice.
- S is an np semilattice if its prime ideal poset is noetherian, i.e. any directed set of prime ideals has a greatest member.
- S is an fp semilattice if each element is a finite meet of primes.
- S is an fpi semilattice if each principal ideal is a finite meet of prime ideals.
- A ring or semilattice S has property M_f if for any $x \in S$ the set of prime ideals not containing x has only finitely many maximal members.

Hyperspectral spaces and their duals

Theorem (ME 2009)

In **ZFC**, *the following statements are equivalent for a space X:*

Hyperspectral spaces and their duals

Theorem (ME 2009)

In **ZFC***, the following statements are equivalent for a space* X*:*

- X *is a hyperspectral space.*

Hyperspectral spaces and their duals

Theorem (ME 2009)

In **ZFC**, *the following statements are equivalent for a space* X:

- X *is a hyperspectral space.*
- X *is a monotone determined spectral space.*

Hyperspectral spaces and their duals

Theorem (ME 2009)

In **ZFC**, *the following statements are equivalent for a space* X:

- X *is a hyperspectral space.*
- X *is a monotone determined spectral space.*
- X *is the Scott space of a quasicoherent domain.*

Hyperspectral spaces and their duals

Theorem (ME 2009)

In **ZFC***, the following statements are equivalent for a space X:*

- X *is a hyperspectral space.*
- X *is a monotone determined spectral space.*
- X *is the Scott space of a quasicoherent domain.*
- X *is the spectrum of a hypercoherent frame.*

Hyperspectral spaces and their duals

Theorem (ME 2009)

In **ZFC***, the following statements are equivalent for a space X:*

- X *is a hyperspectral space.*
- X *is a monotone determined spectral space.*
- X *is the Scott space of a quasicoherent domain.*
- X *is the spectrum of a hypercoherent frame.*
- X *is the prime ideal spectrum of a ring with property* M_f.

Hyperspectral spaces and their duals

Theorem (ME 2009)

In **ZFC***, the following statements are equivalent for a space X:*

- X *is a hyperspectral space.*
- X *is a monotone determined spectral space.*
- X *is the Scott space of a quasicoherent domain.*
- X *is the spectrum of a hypercoherent frame.*
- X *is the prime ideal spectrum of a ring with property M_f.*
- X *is the prime ideal spectrum of a distributive lattice with M_f.*

Hyperspectral spaces and their duals

Theorem (ME 2009)

In **ZFC**, *the following statements are equivalent for a space X:*

- X *is a hyperspectral space.*
- X *is a monotone determined spectral space.*
- X *is the Scott space of a quasicoherent domain.*
- X *is the spectrum of a hypercoherent frame.*
- X *is the prime ideal spectrum of a ring with property* M_f.
- X *is the prime ideal spectrum of a distributive lattice with* M_f.
- X *is the prime filter spectrum of an fpi lattice.*

Duality for quasicoherent domains

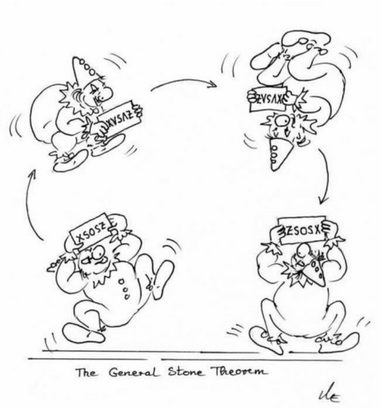
Theorem (ME 2009)

UP *is equivalent to the following statements:*
The category of quasicoherent domains is isomorphic to the category of hyperspectral spaces and to the category of Priestley spaces with the Lawson topology.

These categories are dual to the category of hypercoherent frames (via the open set functor) and to the category of fpi lattices (via Priestley duality).

A similar isomorphism and duality holds for the categories of quasicoherent algebraic domains and superspectral spaces, with supercoherent frames and fp lattices as duals.

Stone-type dualities



Some classical dualities

Theorem

Each of the following dualities is equivalent to **UP***:*

Some classical dualities

Theorem

Each of the following dualities is equivalent to **UP***:*

- *The Hofmann-Lawson duality between δ-continuous frames and locally compact sober spaces.*

Some classical dualities

Theorem

Each of the following dualities is equivalent to **UP***:*

- *The Hofmann-Lawson duality between δ-continuous frames and locally compact sober spaces.*
- *The Hofmann-Lawson-Stralka duality between algebraic frames and Stone spaces (compactly based sober spaces).*

Some classical dualities

Theorem

Each of the following dualities is equivalent to **UP***:*

- *The Hofmann-Lawson duality between δ-continuous frames and locally compact sober spaces.*
- *The Hofmann-Lawson-Stralka duality between algebraic frames and Stone spaces (compactly based sober spaces).*
- *The Grätzer duality between distributive semilattices and Stone spaces.*

Some classical dualities

Theorem

Each of the following dualities is equivalent to **UP***:*

- *The Hofmann-Lawson duality between δ-continuous frames and locally compact sober spaces.*
- *The Hofmann-Lawson-Stralka duality between algebraic frames and Stone spaces (compactly based sober spaces).*
- *The Grätzer duality between distributive semilattices and Stone spaces.*
- *The Stone duality between bounded distributive lattices and spectral spaces.*

Some classical dualities

Theorem

Each of the following dualities is equivalent to **UP***:*

- *The Hofmann-Lawson duality between δ-continuous frames and locally compact sober spaces.*
- *The Hofmann-Lawson-Stralka duality between algebraic frames and Stone spaces (compactly based sober spaces).*
- *The Grätzer duality between distributive semilattices and Stone spaces.*
- *The Stone duality between bounded distributive lattices and spectral spaces.*
- *The Stone duality between boolean algebras and boolean spaces.*

Stone-type dualities

The two categories in the upper row are equivalent
and dual to the isomorphic categories in the lower row

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\substack{\text{finite}\\ \text{frames}}} \arrow[r, shift left, "\substack{\text{identity functor}\\ \text{compact elements}}"] \arrow[d, shift right, "\substack{\text{prime spectrum}\\ \text{with dual order}}"'] & \boxed{\substack{\text{distributive}\\ \text{finite}\\ \text{lattices}}} \arrow[l, shift left, "\text{ideals}"] \arrow[d, shift left, "\substack{\text{prime ideal}\\ \text{spectrum}}"] \\
\boxed{\substack{\text{finite}\\ \text{posets}}} \arrow[u, shift right, "\substack{\text{Scott}\\ \text{open sets}}"'] \arrow[r, shift left, "\text{Scott space}"] & \boxed{\substack{\text{finite } T_0\\ \text{spaces}}} \arrow[l, shift left, "\text{specialization poset}"] \arrow[u, shift left, "\substack{\text{compact}\\ \text{open sets}}"]
\end{tikzcd}
$$

Stone-type dualities

The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\text{superspatial}\\ \text{frames}\end{array}} \arrow[r, shift left, "\textit{compact elements}"] \arrow[d, shift right, "\begin{array}{c}\textit{prime spectrum}\\ \textit{with dual order}\end{array}"'] & \boxed{\begin{array}{c}\text{distributive}\\ \text{np semi-}\\ \text{lattices}\end{array}} \arrow[l, shift left, "\textit{ideals}"] \arrow[d, shift left, "\begin{array}{c}\textit{prime ideal}\\ \textit{spectrum}\end{array}"] \\
\boxed{\begin{array}{c}\text{noetherian}\\ \text{domains}\end{array}} \arrow[u, shift right, "\begin{array}{c}\textit{Scott}\\ \textit{open sets}\end{array}"'] \arrow[r, shift left, "\textit{Scott space}"] & \boxed{\begin{array}{c}\text{Alexandroff}\\ \text{sober}\\ \text{spaces}\end{array}} \arrow[l, shift left, "\textit{specialization poset}"] \arrow[u, shift left, "\begin{array}{c}\textit{compact}\\ \textit{open sets}\end{array}"]
\end{tikzcd}
$$

Stone-type dualities

The two categories in the upper row are equivalent
and dual to the isomorphic categories in the lower row

$$
\begin{tikzcd}
\boxed{\text{superspatial coherent frames}} \arrow[r, shift left, "\text{compact elements}"] \arrow[d, shift right, "\text{prime spectrum with dual order}"'] & \boxed{\text{distributive np lattices}} \arrow[l, shift left, "\text{ideals}"] \arrow[d, shift left, "\text{prime ideal spectrum}"] \\
\boxed{\text{noetherian quasicoherent domains}} \arrow[r, shift left, "\text{Scott space}"] \arrow[u, shift right, "\text{Scott open sets}"'] & \boxed{\text{Alexandroff spectral spaces}} \arrow[l, shift left, "\text{specialization poset}"] \arrow[u, shift left, "\text{compact open sets}"]
\end{tikzcd}
$$

Stone-type dualities

The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{superalgebraic frames}} \arrow[r, shift left, "\textit{compact elements}"] \arrow[d, shift right, "\textit{prime spectrum with dual order}"'] & \boxed{\text{distributive fp semi-lattices}} \arrow[l, shift left, "\textit{ideals}"] \arrow[d, shift left, "\textit{prime ideal spectrum}"] \\
\boxed{\text{algebraic domains}} \arrow[u, shift right, "\textit{Scott open sets}"'] \arrow[r, shift left, "\textit{Scott space}"] & \boxed{\text{supercompactly based sober spaces}} \arrow[l, shift left, "\textit{specialization poset}"] \arrow[u, shift left, "\textit{compact open sets}"]
\end{tikzcd}$$

Stone-type dualities

The two categories in the upper row are equivalent
and dual to the isomorphic categories in the lower row

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{hyperalgebraic frames}} \arrow[r, shift left, "\textit{compact elements}"] \arrow[d, shift right, "\textit{prime spectrum with dual order}"'] & \boxed{\text{distributive fpi semi-lattices}} \arrow[l, shift left, "\textit{ideals}"] \arrow[d, shift left, "\textit{prime ideal spectrum}"] \\
\boxed{\text{quasialgebraic domains}} \arrow[u, shift right, "\textit{Scott open sets}"'] \arrow[r, shift left, "\textit{Scott space}"] & \boxed{\text{hypercompactly based sober spaces}} \arrow[l, shift left, "\textit{specialization poset}"] \arrow[u, shift left, "\textit{compact open sets}"]
\end{tikzcd}
$$

Stone-type dualities

The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row

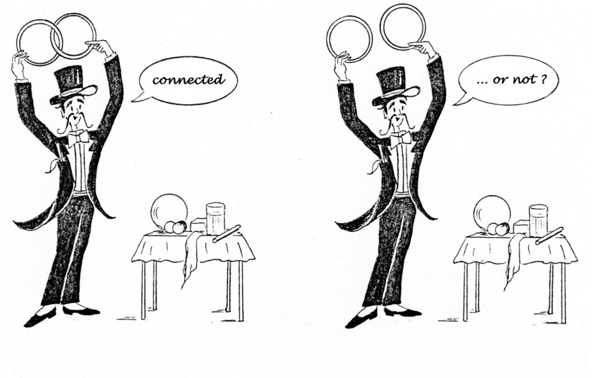
$$
\begin{tikzcd}
\boxed{\text{supercoherent frames}} \arrow[r, shift left, "\textit{compact elements}"] \arrow[d, shift right, "\textit{prime spectrum with dual order}"'] & \boxed{\text{distributive fp lattices}} \arrow[l, shift left, "\textit{ideals}"] \arrow[d, shift left, "\textit{prime ideal spectrum}"] \\
\boxed{\text{Lawson compact algebraic domains}} \arrow[r, shift left, "\textit{Scott space}"] \arrow[u, shift right, "\textit{Scott open sets}"'] & \boxed{\text{superspectral spaces}} \arrow[l, shift left, "\textit{specialization poset}"] \arrow[u, shift left, "\textit{compact open sets}"]
\end{tikzcd}
$$

Stone-type dualities

The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{hypercoherent frames}} \arrow[r, shift left, "\textit{compact elements}"] \arrow[d, shift right, "\textit{prime spectrum with dual order}"'] & \boxed{\text{distributive fpi lattices}} \arrow[l, shift left, "\textit{ideals}"] \arrow[d, shift left, "\textit{prime ideal spectrum}"] \\
\boxed{\text{quasicoherent domains}} \arrow[u, shift right, "\textit{Scott open sets}"'] \arrow[r, shift left, "\textit{Scott space}"] & \boxed{\text{hyperspectral} = \text{spectral Scott spaces}} \arrow[l, shift left, "\textit{specialization poset}"] \arrow[u, shift left, "\textit{compact open sets}"]
\end{tikzcd}$$

Open End: Miraculous topology everywhere



Thanks for your careful attention!