Duality for Effect Algebras and Convex Sets

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20th September 2016

Effect Algebras

- Generalization of Boolean algebras and MV-algebras.
- Also includes orthomodular lattices such as the lattice of closed subspaces of a Hilbert space.
  - Keep $\neg\neg x = x$.
- Structure: $(A, \ovee, \text{-}^{\perp}, 0, 1)$
- $(A, \ovee, 0)$ is a partial commutative monoid:
  - If $a \ovee b$ is defined, $b \ovee a$ is defined and $a \ovee b = b \ovee a$.
  - Associativity is interpreted in a similar way.
  - $a \ovee 0$ is always defined and equals a.

  We write $a \perp b$ to mean $a \ovee b$ is defined.
- $a^{\perp}$ is the unique element such that $a \ovee a^{\perp} = 1$.
- $a \perp 1$ implies $a = 0$.
- Morphisms of effect algebras preserve $\ovee$, where defined, and 0 and 1.

Examples of Effect Algebras

- If $(A, \wedge, \vee, 0, 1)$ is a Boolean algebra, define

$$a \ovee b = \begin{cases} a \vee b & \text{if } a \wedge b = 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

  Then, if we take $a^\perp = \neg a$, $(A, \ovee, \text{-}^\perp, 0, 1)$ is an effect algebra.
- $([0,1], \ovee, \text{-}^\perp, 0, 1)$ is an effect algebra, where

$$a \ovee b = \begin{cases} a + b & \text{if } a + b \leq 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

  and $a^\perp = 1 - a$.
- Effect algebra morphisms $A \to [0,1]$ for A a Boolean algebra are finitely additive probability measures.
- In general maps $A \to [0,1]$ are called states.

Convex Sets

- Abstractly defined as Eilenberg-Moore algebras of $\mathcal{D}$:

$$\mathcal{D}(X) = \left\{ \phi : X \to [0,1] \;\middle|\; \sum_{x \in X} \phi(x) = 1 \right\}$$

  (finitely supported probability distributions, or abstract convex combinations)
- Can be defined in terms of $+_\alpha : X \times X \to X$ for each $\alpha \in [0,1]$, as done by many authors independently.
- We can also define categories of convex subsets of vector spaces, with affine morphisms ignoring the embedding in the vector space.

Dual Adjunction

- We saw $[0,1]$ was an effect algebra. It is also a $\mathcal{D}$-algebra:

$$\alpha : \mathcal{D}([0,1]) \to [0,1] \qquad \alpha(\phi) = \sum_{x \in [0,1]} \phi(x) \cdot x$$

Theorem (Bart Jacobs)

By using $[0,1]$ *as a dualizing object we obtain a dual adjunction:*

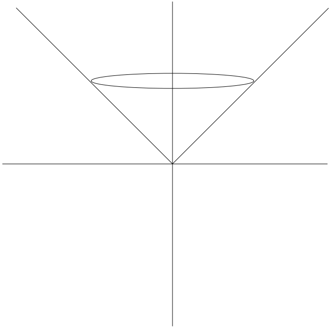
$$\begin{tikzcd} \mathcal{EM}(\mathcal{D}) \arrow[d, shift right, "A=\mathrm{Aff}(-{,}[0{,}1])"'] \arrow[d, phantom, "\dashv"] \\ \mathbf{EA} \arrow[u, shift right, "\mathbf{EA}(-{,}[0{,}1])=S"'] \end{tikzcd}$$

See [Jac10, Theorem 17].

- In every adjunction there is an equivalence.
- What is the duality defined by A and S?

Base-Norm Spaces

- A way of associating a vector space E to a convex set B.



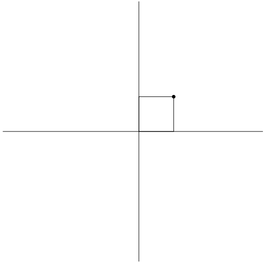
- A norm is defined using the Minkowski functional of $\mathrm{absco}(B)$.
- Not every $\mathcal{EM}(\mathcal{D})$ is embeddable in a vector space.
- Might get a seminorm, but not if B is bounded.

Base-Norm Spaces – Definition

- (E, E_+, τ) – E a real vector space, $E_+ \subseteq E$ a proper cone generating E, $\tau : E \to \mathbb{R}$ a *strictly positive* linear map.
- Positive is $\tau(x) \geq 0$ for all $x \in E_+$, and strictly positive is that if $x \in E_+$ and $\tau(x) = 0$, then $x = 0$.
- Define $B = E_+ \cap \tau^{-1}(1)$. We require $\mathrm{absco}(B)$ to be *radially bounded*. Define $\|\text{-}\|$ to be the Minkowski functional of $\mathrm{absco}(B)$, which is a norm.
- We require E_+ to be $\|\text{-}\|$-closed.
- Inequivalent definitions with the same name are in use, and this fact is never remarked upon.
- Morphisms are linear, positive and preserve τ.

Order-Unit Spaces

- (A, A_+, u), A a real vector space, A_+ a proper cone generating A, u a *strong archimedean unit*.



- Strong unit means that for each $a \in A$, there exists $n \in \mathbb{N}$ such that $-nu \leq a \leq nu$.
- Archimedean means that if $a \leq \frac{1}{n}u$ for all $n \in \mathbb{N}$, then $a \leq 0$.
- Motivating example: self-adjoint part of a C*-algebra, *e.g.* $C(X)$ or $M_n(\mathbb{C})$.
- Morphisms are positive linear maps preserving units.

Relationship to Convex Sets and Effect Algebras

- For $(E, E_+\tau)$ a base-norm space, $B = E_+ \cap \tau^{-1}(1)$ is a convex set, and maps of base-norm spaces restrict to affine maps, making a functor $B : \mathbf{BNS} \to \mathcal{EM}(\mathcal{D})$.
- This is faithful, and also full.
- For (A, A_+, u) an order-unit space, $[0,1]_A$, the unit interval with the addition make partial, is an effect algebra, and maps of order-unit spaces restrict to effect algebra maps, making a functor $[0,1]_{-} : \mathbf{OUS} \to \mathbf{EA}$.
- This is faithful, and (more surprisingly) also full.
- Maps $[0,1]_A \to [0,1]_B$ extend to monotone abelian group homomorphisms $A \to B$. These are continuous and $\mathbb{Q}$-linear, so are $\mathbb{R}$-linear.

Revisiting the Dual Adjunction

- The functor $\mathrm{Aff}(\text{-}, [0,1])$ is $[0,1]_{\text{-}} \circ \mathrm{BAff}(\text{-})$, the set of bounded affine functions to $\mathbb{R}$, an order-unit space.
- The functor $\mathbf{EA}(\text{-}, [0,1])$ is $B \circ S_{\pm}$, where $S_{\pm}$ is the base-norm space of "signed states". (Analogous to signed measures).
- Therefore, if the unit or counit of the duality is an isomorphism, the object in question must be the base of a base-norm space or the unit interval of an order unit space, respectively.
- The unit and counit, when reinterpreted here, are the usual double dual embeddings $E \to E^{**}$ for a normed space.
- Therefore the equivalence defined by A and S is between bases of reflexive base-norm spaces and unit intervals of reflexive order-unit spaces.

Examples of Reflexive Spaces

- All finite-dimensional base-norm and order-unit spaces are reflexive (unless we forgot to require the positive cone to be closed).
- One can form the base-norm space starting with the unit ball of a Hilbert space to get infinite dimensional examples.
- No infinite-dimensional C^*-algebra is reflexive.

Beyond Reflexiveness

- There is a way to make every Banach space "reflexive", using weak topologies.
- If E^* is given the weak-* topology, the embedding $E \to (E^*)'$ is an isomorphism.
- Dual spaces can be characterized as having compact unit balls. (A sort of converse to Banach-Alaoglu).[Ng71]
- One can characterize the *bounded weak-** topology, Smith spaces. [Akb09]

Smith Base-Norm and Order-Unit Spaces

- We can then get two different dualities depending on whether we put the weak topology on the order-unit spaces or the base-norm spaces.
- For this purpose we can define Smith base-norm spaces and Smith order-unit spaces, where the base and unit interval, respectively, are required to be compact (similarly to a characterization by Ellis [Ell64]).

$$\mathbf{SBNS} \simeq \mathbf{BOUS}^{\mathrm{op}} \qquad \mathbf{BBNS} \simeq \mathbf{SOUS}^{\mathrm{op}}$$

References

- Sergei S. Akbarov, *Holomorphic Functions of Exponential Type and Duality for Stein Groups with Algebraic Connected Component of Identity*, Journal of Mathematical Sciences **162** (2009), no. 4, 459–586 (English).
- Alan J. Ellis, *The Duality of Partially Ordered Normed Linear Spaces*, Journal of the London Mathematical Society **s1-39** (1964), no. 1, 730–744.
- Bart Jacobs, *Convexity, Duality and Effects*, Theoretical Computer Science (Cristian S. Calude and Vladimiro Sassone, eds.), IFIP Advances in Information and Communication Technology, vol. 323, Springer Berlin Heidelberg, 2010, pp. 1–19.
- Kung-fu Ng, *On a Theorem of Dixmier*, Mathematica Scandinavica **29** (1971), 279–280.