Hausdorff invariance theorems for localic maps, and their duals

Jorge Picado <picado@mat.uc.pt>

— joint work with J. Gutiérrez García and T. Kubiak

Centre for Mathematics
University of Coimbra

MOTTO

(HAUSDORFF) Mapping Invariance Theorem

Let $f\colon X \to Y$ be a CLOSED surjection.

If X is normal then Y is also normal.

(*Fund. Math.* (1935))

GOAL: pointfree mapping invariance theorems

CLASSICAL TOPOLOGY $\rightsquigarrow$ POINTFREE TOPOLOGY

topological spaces

generalized spaces:
locales

GOAL: pointfree mapping invariance theorems

CLASSICAL TOPOLOGY $\rightsquigarrow$ POINTFREE TOPOLOGY

topological spaces

generalized spaces: locales

CABOOL

SUBOBJ. LATTICES

COFRAME

GOAL: pointfree mapping invariance theorems

CLASSICAL TOPOLOGY ⇝ POINTFREE TOPOLOGY

topological spaces

generalized spaces: locales

CABOOL | SUBOBJ. LATTICES | COFRAME

«(...) a locale has enough complemented sublocales to compensate for this shortcoming: one simply has to make the sublocales which are complemented do more of the work.»

JOHN ISBELL

(Atomless parts of spaces, *Math. Scand.* (1972))

The category of locales CONCRETELY

OBJECTS: locales=frames

Complete lattices satisfying $\boxed{a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)}$

The category of locales CONCRETELY

OBJECTS: locales=frames

Complete lattices satisfying $\boxed{a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)}$

(= complete Heyting algebras)

The category of locales CONCRETELY

OBJECTS: locales=frames

Complete lattices satisfying $\boxed{a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)}$

(= complete Heyting algebras)

MORPHISMS: localic maps

$f\colon L \longrightarrow M$

- $f(\bigwedge S) = \bigwedge f[S]$

The category of locales CONCRETELY

OBJECTS: locales=frames

Complete lattices satisfying $\boxed{a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)}$

(= complete Heyting algebras)

MORPHISMS: localic maps

$f\colon L \longrightarrow M$

f^*

- $f(\bigwedge S) = \bigwedge f[S]$

The category of locales CONCRETELY

OBJECTS: locales=frames

Complete lattices satisfying $\boxed{a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)}$

(= complete Heyting algebras)

MORPHISMS: localic maps

$$
\begin{tikzcd}
L \arrow[r, "f"] & M \arrow[l, bend left, "f^*"]
\end{tikzcd}
$$

- $f(\bigwedge S) = \bigwedge f[S]$
- $f(a) = 1 \Rightarrow a = 1$

The category of locales CONCRETELY

OBJECTS: locales=frames

Complete lattices satisfying $\boxed{a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)}$

(= complete Heyting algebras)

MORPHISMS: localic maps

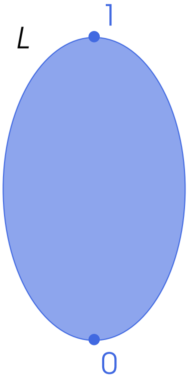
$f\colon L \longrightarrow M$

f^*

- $f(\bigwedge S) = \bigwedge f[S]$
- $f(a) = 1 \Rightarrow a = 1$
- $f(f^*(a) \to b) = a \to f(b)$

TOOLS: sublocales

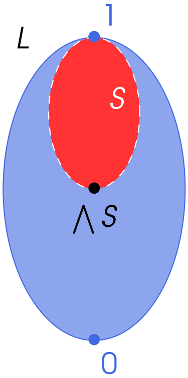
$S \subseteq L$ is a SUBLOCALE of L if:



TOOLS: sublocales

$S \subseteq L$ is a SUBLOCALE of L if:

(1) $\forall A \subseteq S$, $\bigwedge A \in S$.

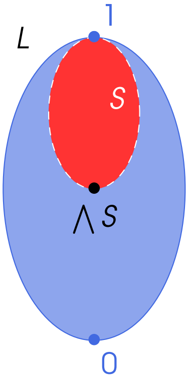


TOOLS: sublocales

$S \subseteq L$ is a SUBLOCALE of L if:

(1) $\forall A \subseteq S,\ \bigwedge A \in S$.

(2) $\forall a \in L, \forall s \in S,\ a \to s \in S$.



TOOLS: sublocales

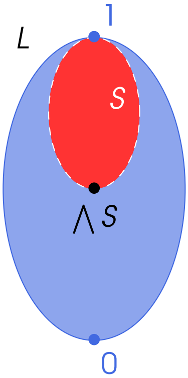
$S \subseteq L$ is a SUBLOCALE of L if:

(1) $\forall A \subseteq S,\ \bigwedge A \in S$.

(2) $\forall a \in L, \forall s \in S,\ a \to s \in S$.

S is itself a locale: $\bigwedge_S = \bigwedge_L,\ \to_S = \to_L$

but $\bigsqcup s_i = \bigwedge\{s \in S \mid \bigvee s_i \leqslant s\}$.

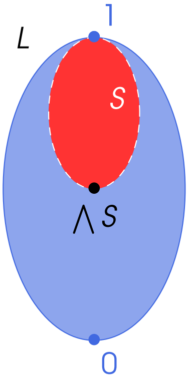


TOOLS: sublocales

$S \subseteq L$ is a SUBLOCALE of L if:

(1) $\forall A \subseteq S,\ \bigwedge A \in S$.

(2) $\forall a \in L, \forall s \in S,\ a \to s \in S$.



S is itself a locale: $\bigwedge_S = \bigwedge_L,\ \to_S = \to_L$

but $\bigsqcup s_i = \bigwedge\{s \in S \mid \bigvee s_i \leqslant s\}$.

Motivation for the definition:

Proposition

$S \subseteq L$ is a sublocale iff the embedding $j_S \colon S \subseteq L$ is a localic map.

TOOLS: sublocale lattices

sublocales of L, ordered by $\subseteq$:

TOOLS: sublocale lattices

sublocales of L, ordered by $\subseteq$:

$$\mathbf{0} = \{1\}, \quad \mathbf{1} = L, \quad \bigwedge = \bigcap, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

TOOLS: sublocale lattices

sublocales of L, ordered by $\subseteq$:

$$\mathbf{0} = \{1\}, \quad \mathbf{1} = L, \quad \bigwedge = \bigcap, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

Proposition:
This lattice is a coframe.

TOOLS: sublocale lattices

sublocales of L, ordered by $\subseteq$:

$$\mathbf{0} = \{1\}, \quad \mathbf{1} = L, \quad \bigwedge = \bigcap, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

Proposition:
This lattice is a coframe.

Special sublocales:

$a \in L, \qquad \mathfrak{c}(a) = {\uparrow} a$ CLOSED

TOOLS: sublocale lattices

sublocales of L, ordered by $\subseteq$:

$$\mathbf{0} = \{1\}, \quad \mathbf{1} = L, \quad \bigwedge = \bigcap, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

Proposition:
This lattice is a coframe.

Special sublocales:

$a \in L, \qquad \mathfrak{c}(a) = {\uparrow} a \qquad$ CLOSED

$\qquad \mathfrak{o}(a) = \{a \to x \mid x \in L\} \qquad$ OPEN

TOOLS: sublocale lattices

sublocales of L, ordered by $\subseteq$:

$$\mathbf{0} = \{1\}, \quad \mathbf{1} = L, \quad \bigwedge = \bigcap, \quad \bigvee_I S_i = \{\bigwedge A \mid A \subseteq \bigcup_I S_i\}$$

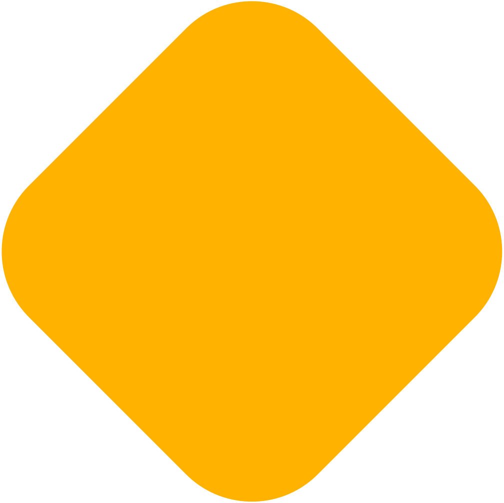
Proposition:
This lattice is a coframe.

Special sublocales:

$$\left.\begin{array}{lll} a \in L, & \mathfrak{c}(a) = {\uparrow}a & \text{CLOSED} \\ & \mathfrak{o}(a) = \{a \to x \mid x \in L\} & \text{OPEN} \end{array}\right\} \text{complemented}$$

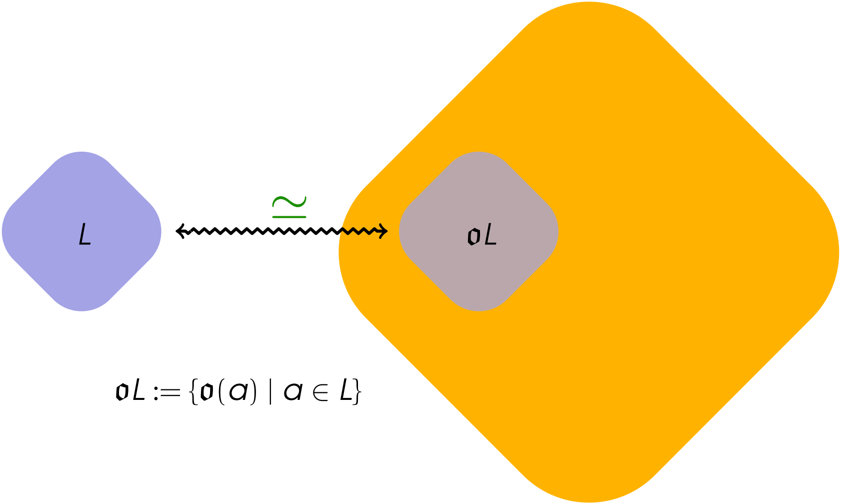
TOOLS: sublocale lattices

coframe of sublocales



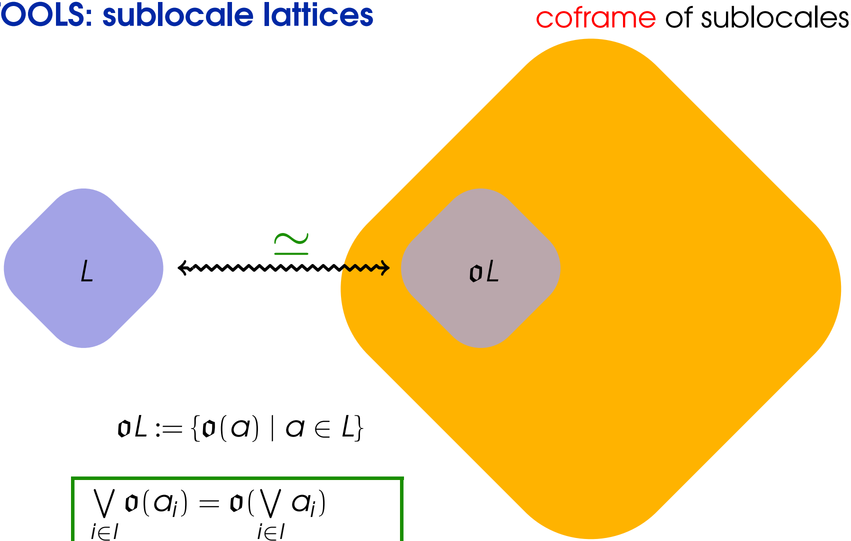
TOOLS: sublocale lattices

coframe of sublocales



$$\mathfrak{o}L := \{\mathfrak{o}(a) \mid a \in L\}$$

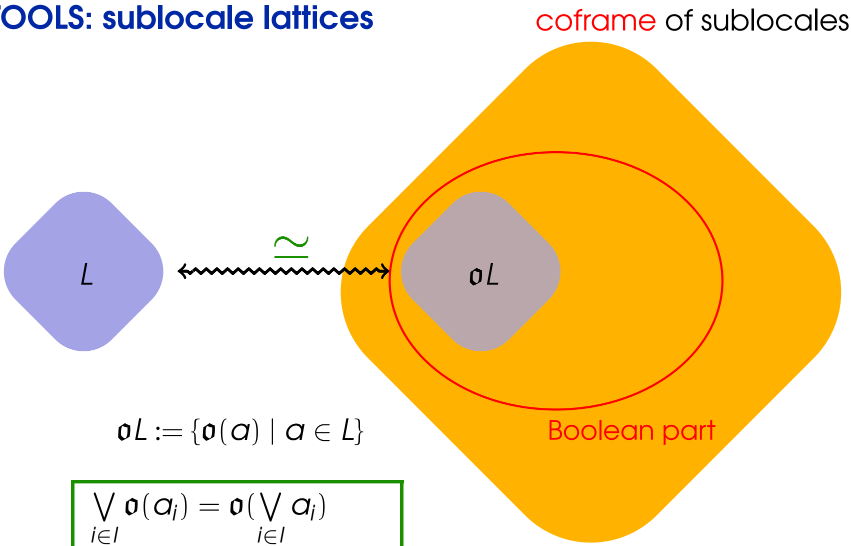
TOOLS: sublocale lattices



$$\mathfrak{o}L := \{\mathfrak{o}(a) \mid a \in L\}$$

$$\boxed{\begin{array}{l} \bigvee_{i\in I} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i\in I} a_i) \\ \mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b) \end{array}}$$

TOOLS: sublocale lattices

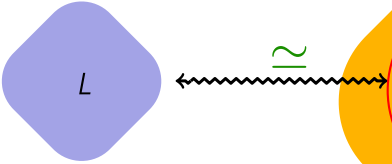


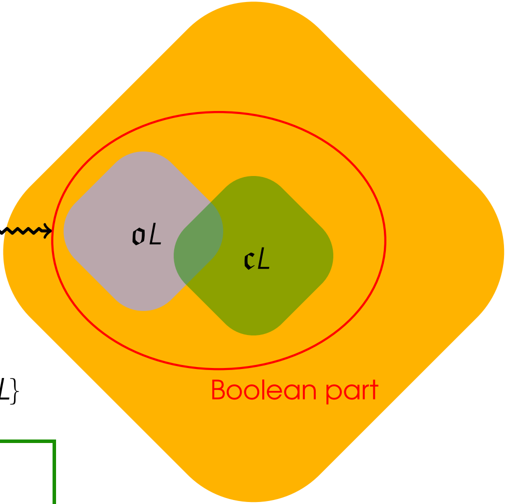
$$\mathfrak{o}L := \{\mathfrak{o}(a) \mid a \in L\}$$

$$\boxed{\begin{array}{l} \bigvee_{i\in I} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i\in I} a_i) \\ \mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b) \end{array}}$$

TOOLS: sublocale lattices

coframe of sublocales

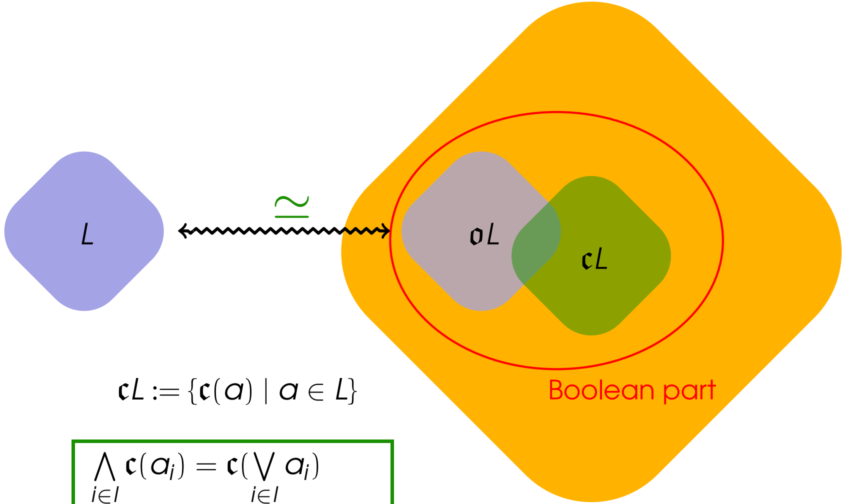




$$\mathfrak{c}L := \{\mathfrak{c}(a) \mid a \in L\}$$

$$\boxed{\begin{gathered}\bigwedge_{i\in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i\in I} a_i) \\ \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)\end{gathered}}$$

TOOLS: sublocale lattices



$$\mathfrak{c}L := \{\mathfrak{c}(a) \mid a \in L\}$$

$$\boxed{\begin{array}{l} \bigwedge_{i\in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i\in I} a_i) \\ \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b) \end{array}}$$

TOOLS: images

localic map $\begin{array}{l} f\colon L \longrightarrow M \\ \ \cup\!| \\ \ S \end{array}$

TOOLS: images

localic map $f\colon L \longrightarrow M$

$$\begin{array}{c} \cup\!| \\ S \end{array}$$

$$\begin{tikzcd} L \arrow[r, "f"] & M \\ S \arrow[u, hook, "j_S"] & \end{tikzcd}$$

TOOLS: images

localic map $f\colon L \longrightarrow M$
$$\begin{array}{c} \cup\!\shortmid \\ S \end{array}$$

$$\begin{tikzcd} L \arrow[r, "f"] & M \\ S \arrow[u, hook, "j_S"] \arrow[r, dotted, "\text{epi}"'] & f[S] \arrow[u, dotted, hook, "\text{regular mono}"'] \end{tikzcd}$$

TOOLS: images

localic map $f\colon L \longrightarrow M$

$$\begin{array}{c} \cup\!\mid \\ S \end{array}$$

$$\begin{tikzcd} L \arrow[r, "f"] & M \\ S \arrow[u, hook, "j_S"] \arrow[r, dotted, "\text{epi}"'] & f[S] \arrow[u, dotted, hook, "\text{regular mono}"'] \end{tikzcd}$$

$f[S]$ is a sublocale of M

the image of S under f

TOOLS: images

localic map $f\colon L \longrightarrow M$
$\cup\!|$
S

$$\begin{tikzcd} L \arrow[r, "f"] & M \\ S \arrow[u, hook, "j_S"] \arrow[r, dotted, "\text{epi}"'] & f[S] \arrow[u, hook, dotted, "\text{regular mono}"'] \end{tikzcd}$$

is a sublocale of M

the image of S under f

IMAGE MAP: $f[-]\colon \mathcal{S}(L) \to \mathcal{S}(M)$ (localic map)

TOOLS: images

localic map $f\colon \begin{array}[t]{c} L \\ \cup\vert \\ S \end{array} \longrightarrow M$

$$\begin{tikzcd}
L \arrow[r, "f"] & M \\
S \arrow[u, hook, "j_S"] \arrow[r, dotted, "epi"'] & f[S] \arrow[u, hook, dotted, "\text{regular mono}"']
\end{tikzcd}$$

$f[S]$ is a sublocale of M

$\boxed{\text{the image of } S \text{ under } f}$

IMAGE MAP: $f[-]\colon \mathcal{S}(L) \to \mathcal{S}(M)$ (localic map)

CLOSED MAP: $f[S]$ is closed for every closed S

TOOLS: images

localic map $f\colon L \longrightarrow M$
$\cup\!\mid$
S

$$\begin{tikzcd} L \arrow[r, "f"] & M \\ S \arrow[u, hook, "j_S"] \arrow[r, dotted, "\text{epi}"'] & f[S] \arrow[u, dotted, hook, "\text{regular mono}"'] \end{tikzcd}$$

is a sublocale of M

the image of S under f

IMAGE MAP: $f[-]\colon \mathcal{S}(L) \to \mathcal{S}(M)$ (localic map)

CLOSED MAP: $f[S]$ is closed for every closed S

$$\Leftrightarrow \quad f[\mathfrak{c}(a)] = \mathfrak{c}(f(a)) \quad \forall a \in L$$

TOOLS: preimages

localic map $f\colon L \longrightarrow M$

$$\begin{array}{r} f\colon L \longrightarrow M \\ \cup\!\!\mid \\ T \end{array}$$

TOOLS: preimages

localic map $f\colon L \longrightarrow M$

$$\begin{array}{r} M \\ \cup\!\mid \\ T \end{array}$$

for any $A \subseteq L$ closed under meets: $\{1\} \subseteq A$

$$S_i \subseteq A \Rightarrow \bigvee S_i \subseteq A$$

TOOLS: preimages

localic map $f \colon L \longrightarrow M$

$$\begin{array}{r} M \\ \cup\!\mid \\ T \end{array}$$

for any $A \subseteq L$ closed under meets: $\{1\} \subseteq A$

$$S_i \subseteq A \Rightarrow \underbrace{\bigvee S_i}_{\{\bigwedge B \mid B \subseteq \bigcup S_i\}} \subseteq A$$

TOOLS: preimages

localic map $f: L \longrightarrow M$
$$\begin{array}{r} M \\ \cup\!\!\mid \\ T \end{array}$$

for any $A \subseteq L$ closed under meets: $\{1\} \subseteq A$

$$S_i \subseteq A \Rightarrow \underbrace{\bigvee S_i}_{\{\bigwedge B \mid B \subseteq \bigcup S_i\}} \subseteq A$$

So there is the largest sublocale contained in A: A_{sloc}

TOOLS: preimages

localic map $f\colon L \longrightarrow M$

$$M \supseteq T$$

for any $A \subseteq L$ closed under meets: $\{1\} \subseteq A$

$$S_i \subseteq A \Rightarrow \underbrace{\bigvee S_i}_{\{\bigwedge B \mid B \subseteq \bigcup S_i\}} \subseteq A$$

So there is the largest sublocale contained in A: A_{sloc}

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ & & \cup\!\mid \\ f^{-1}[T] & & T \end{array}$$

TOOLS: preimages

localic map $f\colon L \longrightarrow M$

$$M \supseteq T$$

for any $A \subseteq L$ closed under meets: $\{1\} \subseteq A$

$$S_i \subseteq A \Rightarrow \underbrace{\bigvee S_i}_{\{\bigwedge B \mid B \subseteq \bigcup S_i\}} \subseteq A$$

So there is the largest sublocale contained in A: A_{sloc}

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ & & \cup\!| \\ \underbrace{f^{-1}[T]} & & T \end{array}$$

closed under meets (since f preserve meets)

TOOLS: preimages

localic map $f\colon L \longrightarrow M \supseteq T$

for any $A \subseteq L$ closed under meets: $\{1\} \subseteq A$

$$S_i \subseteq A \Rightarrow \underbrace{\bigvee S_i}_{\{\bigwedge B \mid B \subseteq \bigcup S_i\}} \subseteq A$$

So there is the largest sublocale contained in A: A_{sloc}

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ & & \cup\!| \\ \underbrace{f^{-1}[T]} & & T \end{array}$$

$$f_{-1}[T] = (f^{-1}[T])_{\text{sloc}}$$

the preimage of T under f

closed under meets (since f preserve meets)

TOOLS: preimages

localic map $f\colon L \longrightarrow M$
$\phantom{f\colon L \longrightarrow{}} \cup\!\!\mid$
$\phantom{f\colon L \longrightarrow{}} T$

for any $A \subseteq L$ closed under meets: $\{1\} \subseteq A$

$$S_i \subseteq A \Rightarrow \underbrace{\bigvee S_i}_{\{\bigwedge B \mid B \subseteq \bigcup S_i\}} \subseteq A$$

So there is the largest sublocale contained in A: A_{sloc}

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ & & \cup\!\!\mid \\ \underbrace{f^{-1}[T]} & & T \end{array}$$

closed under meets (since f preserve meets)

$$f_{-1}[T] = (f^{-1}[T])_{\mathrm{sloc}}$$

the preimage of T under f

PREIMAGE MAP: $f_{-1}[-]\colon \mathcal{S}(M) \to \mathcal{S}(L)$ (frame homom.)

TOOLS: images and preimages

$f_{-1}[-] \dashv f[-]$ AS IT SHOULD BE!

TOOLS: images and preimages

$f_{-1}[-] \dashv f[-]$ AS IT SHOULD BE!

PROPERTIES

1. $f_{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a))$ and $f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$.

TOOLS: images and preimages

$f_{-1}[-] \dashv f[-]$ AS IT SHOULD BE!

PROPERTIES

1. $f_{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a))$ and $f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$.
2. $f_{-1}[-]$ preserves complements.

TOOLS: images and preimages

$f_{-1}[-] \dashv f[-]$

AS IT SHOULD BE!

PROPERTIES

1. $f_{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a))$ and $f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$.

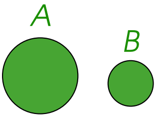
2. $f_{-1}[-]$ preserves complements.

3. for surjective f: $f\, f_{-1}[\mathfrak{c}(a)] = \mathfrak{c}(a)$ and $f\, f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(a)$.

Doing topology in Loc

Normality

$\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1$



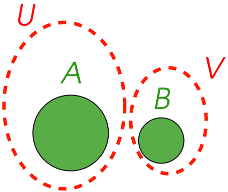
Doing topology in Loc

Normality

$$\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1$$

$$\Downarrow$$

$$\exists u, v\colon\ \mathfrak{o}(u) \vee \mathfrak{o}(v) = 1,\ \mathfrak{c}(a) \geqslant \mathfrak{o}(u),\ \mathfrak{c}(b) \geqslant \mathfrak{o}(v).$$

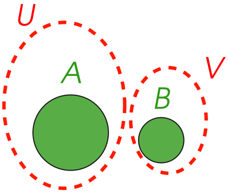


Doing topology in Loc

$\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1$

$\Downarrow$

$\exists u, v\colon\ \mathfrak{o}(u) \vee \mathfrak{o}(v) = 1,\ \mathfrak{c}(a) \geqslant \mathfrak{o}(u),\ \mathfrak{c}(b) \geqslant \mathfrak{o}(v).$



So *L* is normal iff

$$\boxed{\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1 \ \Rightarrow\ \exists u, v\colon\ \mathfrak{c}(u) \wedge \mathfrak{c}(v) = 0,\ \mathfrak{c}(a) \vee \mathfrak{c}(u) = 1 = \mathfrak{c}(b) \vee \mathfrak{c}(v)}$$

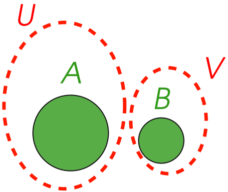
Doing topology in Loc

Normality

$\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1$

$\Downarrow$

$\exists u, v\colon\ \mathfrak{o}(u) \vee \mathfrak{o}(v) = 1,\ \mathfrak{c}(a) \geqslant \mathfrak{o}(u),\ \mathfrak{c}(b) \geqslant \mathfrak{o}(v).$



So *L* is normal iff

$$\boxed{\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1 \ \Rightarrow\ \exists u, v\colon\ \mathfrak{c}(u) \wedge \mathfrak{c}(v) = 0,\ \mathfrak{c}(a) \vee \mathfrak{c}(u) = 1 = \mathfrak{c}(b) \vee \mathfrak{c}(v)}$$

Internally in L:
(by $\mathfrak{c}L \cong L$)

$$\boxed{a \vee b = 1 \ \Rightarrow \exists u, v\colon\ u \wedge v = 0,\ a \vee u = 1 = b \vee v}$$

(Conservative extension: X is normal iff the locale $\mathcal{O}(X)$ is normal.)

THE INVARIANCE THEOREM: first version

Theorem

Let $f\colon L \to M$ be a CLOSED surjective localic map.

If L is normal then M is also normal.

THE INVARIANCE THEOREM: first version

Theorem

Let $f\colon L \to M$ be a CLOSED surjective localic map.

If L is normal then M is also normal.

Proof.

Later on ...

AIM I: to cover other variants of normality

Boolean sublocale selections

$\mathscr{B}\colon L \mapsto \mathscr{B}(L) \subseteq B(\mathcal{S}(L))$ "sets of complemented sublocales"

Boolean sublocale selections

$\mathscr{B}\colon L \mapsto \mathscr{B}(L) \subseteq B(\mathcal{S}(L))$ "sets of complemented sublocales"

Selection $\mathscr{B}$	Members of $\mathscr{B}(L)$
$\mathfrak{c}$	$\{\mathfrak{c}(a)\colon a \in L\}$

the standard model

Boolean sublocale selections

$\mathscr{B}\colon L \mapsto \mathscr{B}(L) \subseteq B(\mathcal{S}(L))$ "sets of complemented sublocales"

Selection $\mathscr{B}$	Members of $\mathscr{B}(L)$
$\mathfrak{c}$	$\{\mathfrak{c}(a)\colon a \in L\}$
$\mathfrak{c}^*$	$\{\mathfrak{c}(a^*)\colon a \in L\}$

Boolean sublocale selections

$\mathscr{B}\colon L \mapsto \mathscr{B}(L) \subseteq B(\mathcal{S}(L))$ "sets of complemented sublocales"

Selection $\mathscr{B}$	Members of $\mathscr{B}(L)$
$\mathfrak{c}$	$\{\mathfrak{c}(a)\colon a \in L\}$
$\mathfrak{c}^*$	$\{\mathfrak{c}(a^*)\colon a \in L\}$
$\mathfrak{c}_\delta$	$\{\mathfrak{c}(a)\colon a \text{ is regular } G_\delta\}$

regular G_δ element: $a = \bigvee_{n\in\mathbb{N}} a_n$ with $a_n \prec a$

Boolean sublocale selections

$\mathscr{B}\colon L \mapsto \mathscr{B}(L) \subseteq B(\mathcal{S}(L))$ "sets of complemented sublocales"

Selection $\mathscr{B}$	Members of $\mathscr{B}(L)$
$\mathfrak{c}$	$\{\mathfrak{c}(a)\colon a \in L\}$
$\mathfrak{c}^*$	$\{\mathfrak{c}(a^*)\colon a \in L\}$
$\mathfrak{c}_\delta$	$\{\mathfrak{c}(a)\colon a$ is regular $G_\delta\}$
$\mathfrak{c}_{\mathrm{coz}}$	$\{\mathfrak{c}(\mathrm{coz}\, f)\colon f \in \mathrm{C}(L)\}$

regular G_δ element: $a = \bigvee_{n\in\mathbb{N}} a_n$ with $a_n \prec a$

cozero element: $\boxed{a = \bigvee_{n\in\mathbb{N}} a_n \text{ with } a_n \prec\!\!\prec a}$

Boolean sublocale selections

$\mathscr{B}\colon L \mapsto \mathscr{B}(L) \subseteq B(\mathcal{S}(L))$ "sets of complemented sublocales"

Selection $\mathscr{B}$	**Members of $\mathscr{B}(L)$**
$\mathfrak{c}$	$\{\mathfrak{c}(a)\colon a \in L\}$
$\mathfrak{c}^*$	$\{\mathfrak{c}(a^*)\colon a \in L\}$
$\mathfrak{c}_\delta$	$\{\mathfrak{c}(a)\colon a$ is regular $G_\delta\}$
$\mathfrak{c}_{\mathrm{coz}}$	$\{\mathfrak{c}(\mathrm{coz}\, f)\colon f \in \mathrm{C}(L)\}$

J. Gutiérrez García & JP, *On the parallel between normality and extremal disconnectedness*, JPAA 218 (2014) 784-803

$\mathscr{B}$-NORMALITY

Normal:

$$\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1 \Rightarrow \exists u, v\colon\ \mathfrak{c}(u) \wedge \mathfrak{c}(v) = 0,\ \mathfrak{c}(a) \vee \mathfrak{c}(u) = 1 = \mathfrak{c}(b) \vee \mathfrak{c}(v).$$

$\mathscr{B}$-NORMALITY

L is $\mathscr{B}$-Normal (for a fixed sublocale selection $\mathscr{B}$):

For any $A, B \in \mathscr{B}(L)$,

$$\boxed{A \vee B = 1 \ \Rightarrow \ \exists\, U, V \in \mathscr{B}(L):\ U \wedge V = 0,\ A \vee U = 1 = B \vee V}$$

Normal:

$$\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1 \ \Rightarrow\ \exists u, v:\ \mathfrak{c}(u) \wedge \mathfrak{c}(v) = 0,\ \mathfrak{c}(a) \vee \mathfrak{c}(u) = 1 = \mathfrak{c}(b) \vee \mathfrak{c}(v).$$

$\mathscr{B}$-NORMALITY

L is $\mathscr{B}$-Normal (for a fixed sublocale selection $\mathscr{B}$):

For any $A, B \in \mathscr{B}(L)$,

$$\boxed{A \vee B = 1 \;\; \Rightarrow \;\; \exists\, U, V \in \mathscr{B}(L): \; U \wedge V = 0, \; A \vee U = 1 = B \vee V}$$

Selection $\mathscr{B}$	**$\mathscr{B}$-normal frames**
$\mathfrak{c}$	normal

$\mathscr{B}$-NORMALITY

L is $\mathscr{B}$-Normal (for a fixed sublocale selection $\mathscr{B}$):

For any $A, B \in \mathscr{B}(L)$,

$$\boxed{A \vee B = 1 \;\Rightarrow\; \exists\, U, V \in \mathscr{B}(L):\; U \wedge V = 0,\; A \vee U = 1 = B \vee V}$$

Selection $\mathscr{B}$	$\mathscr{B}$-normal frames
$\mathfrak{c}$	normal
$\mathfrak{c}^*$	mildly normal

$\mathscr{B}$-NORMALITY

L is $\mathscr{B}$-Normal (for a fixed sublocale selection $\mathscr{B}$):

For any $A, B \in \mathscr{B}(L)$,

$$\boxed{A \vee B = 1 \quad \Rightarrow \quad \exists\, U, V \in \mathscr{B}(L): \ U \wedge V = 0, \ A \vee U = 1 = B \vee V}$$

Selection $\mathscr{B}$	**$\mathscr{B}$-normal frames**
$\mathfrak{c}$	normal
$\mathfrak{c}^*$	mildly normal
$\mathfrak{c}_\delta$	δ-normal

$\mathscr{B}$-NORMALITY

L is $\mathscr{B}$-Normal (for a fixed sublocale selection $\mathscr{B}$):

For any $A, B \in \mathscr{B}(L)$,

$$\boxed{A \vee B = 1 \ \Rightarrow\ \exists\, U, V \in \mathscr{B}(L):\ U \wedge V = 0,\ A \vee U = 1 = B \vee V}$$

Selection $\mathscr{B}$	**$\mathscr{B}$-normal frames**
$\mathfrak{c}$	normal
$\mathfrak{c}^*$	mildly normal
$\mathfrak{c}_\delta$	δ-normal
$\mathfrak{c}_{\mathrm{coz}}$	all frames

The Invariance Theorem: 1st version

localic map $f\colon L \to M$

Theorem

Let $f\colon L \to M$ be a CLOSED surjective localic map.

If L is normal then M is also normal.

The Invariance Theorem: 1st version

localic map $f\colon L \to M$

f is image $\mathscr{B}$-preserving if

$f[-]$ maps elements of $\mathscr{B}(L)$ into $\mathscr{B}(M)$.

Theorem

Let $f\colon L \to M$ be a CLOSED surjective localic map.

If L is normal then M is also normal.

The Invariance Theorem: 1st version

localic map $f\colon L \to M$

f is image $\mathscr{B}$-preserving if

$f[-]$ maps elements of $\mathscr{B}(L)$ into $\mathscr{B}(M)$.

f is preimage $\mathscr{B}$-preserving if

$f_{-1}[-]$ maps elements of $\mathscr{B}(M)$ into $\mathscr{B}(L)$.

Theorem

Let $f\colon L \to M$ be a CLOSED surjective localic map.

If L is normal then M is also normal.

The Invariance Theorem: general version localic map $f\colon L \to M$

f is image $\mathscr{B}$-preserving if

$f[-]$ maps elements of $\mathscr{B}(L)$ into $\mathscr{B}(M)$.

f is preimage $\mathscr{B}$-preserving if

$f_{-1}[-]$ maps elements of $\mathscr{B}(M)$ into $\mathscr{B}(L)$.

Theorem

Let $f\colon L \to M$ be a image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving surjective localic map.

If L is $\mathscr{B}$-normal then M is also $\mathscr{B}$-normal.

PROOF:

$$L \xrightarrow{f} M$$

$$A,\ B \in \mathscr{B}(M), \quad A \vee B = 1$$

PROOF:

$$
\begin{tikzcd}
L \arrow[r, "f"] & M
\end{tikzcd}
$$

$$
\begin{array}{c}
A,\ B \in \mathscr{B}(M), \quad A \vee B = 1 \\
\downarrow \quad \downarrow \quad \text{hyp.} \\
f_{-1}[A], f_{-1}[B] \ \in \mathscr{B}(L)
\end{array}
$$

PROOF:

$$L \xrightarrow{f} M$$

$$A,\ B \in \mathscr{B}(M), \quad A \vee B = 1$$

$$\Big\downarrow \text{ hyp.}$$

$$f_{-1}[A] \vee f_{-1}[B] = f_{-1}[1] = 1 \qquad\qquad f_{-1}[A], f_{-1}[B] \ \in \mathscr{B}(L)$$

PROOF:

$$L \xrightarrow{f} M$$

$A,\ B \in \mathscr{B}(M),\quad A \vee B = 1$

hyp.

$f_{-1}[A], f_{-1}[B]\ \in \mathscr{B}(L)$

$f_{-1}[A] \vee f_{-1}[B] = f_{-1}[1] = 1$

$\Downarrow$ L is $\mathscr{B}$-normal

$\exists\ U_0,\ V_0 \in \mathscr{B}(L):\ U_0 \wedge V_0 = 0,\quad f_{-1}[A] \vee U_0 = 1 = f_{-1}[B] \vee V_0.$

hyp.

PROOF: $L \xrightarrow{\ f\ } M$

$$A,\ B \in \mathscr{B}(M), \quad A \vee B = 1$$

hyp.

$$f_{-1}[A],\ f_{-1}[B] \ \in \mathscr{B}(L)$$

$$f_{-1}[A] \vee f_{-1}[B] = f_{-1}[1] = 1$$

$$\Downarrow \quad L \text{ is } \mathscr{B}\text{-normal}$$

$$\exists\ U_0,\ V_0 \in \mathscr{B}(L):\ \ U_0 \wedge V_0 = 0, \quad f_{-1}[A] \vee U_0 = 1 = f_{-1}[B] \vee V_0.$$

hyp.

$U = f[U_0]$, $V = f[V_0] \in \mathscr{B}(M)$ satisfy:

PROOF: $L \xrightarrow{f} M$

$A,\ B \in \mathscr{B}(M),\quad A \vee B = 1$

hyp.

$f_{-1}[A], f_{-1}[B] \ \in \mathscr{B}(L)$

$f_{-1}[A] \vee f_{-1}[B] = f_{-1}[1] = 1$

$\Downarrow$ L is $\mathscr{B}$-normal

$\exists\ U_0,\ V_0 \in \mathscr{B}(L):\ \ U_0 \wedge V_0 = 0,\quad f_{-1}[A] \vee U_0 = 1 = f_{-1}[B] \vee V_0.$

hyp.

$U = f[U_0],\ V = f[V_0] \in \mathscr{B}(M)$ satisfy:

hyp.

- $U \wedge V = f[U_0] \wedge f[V_0] = f[U_0 \wedge V_0] = f[0] = f[L] = M = 0.$

PROOF:

$$L \xrightarrow{f} M$$

$$A,\ B \in \mathscr{B}(M), \quad A \vee B = 1$$

hyp.

$$f_{-1}[A] \vee f_{-1}[B] = f_{-1}[1] = 1 \qquad\qquad f_{-1}[A], f_{-1}[B] \ \in \mathscr{B}(L)$$

$$\Downarrow$$

L is $\mathscr{B}$-normal

$$\exists\ U_0,\ V_0 \in \mathscr{B}(L):\ U_0 \wedge V_0 = 0,\ \boxed{f_{-1}[A] \vee U_0 = 1} = f_{-1}[B] \vee V_0.$$

hyp.

$$U = f[U_0],\ V = f[V_0] \in \mathscr{B}(M)$$ satisfy:

hyp.

- $U \wedge V = f[U_0] \wedge f[V_0] = f[U_0 \wedge V_0] = f[0] = f[L] = M = 0.$
- $U = f[U_0] \geqslant f\, f_{-1}[A^c] \geqslant A^c$, i.e. $A \vee U = 1$ (and similarly for V).

$f_{-1}[-]$ preserves complements

Image and preimage $\mathscr{B}$-preserving maps $f\colon L \to M$

image $\mathscr{B}$-preserving: $f[-]$ maps elements of $\mathscr{B}(L)$ into $\mathscr{B}(M)$.

preimage $\mathscr{B}$-preserving: $f_{-1}[-]$ maps elements of $\mathscr{B}(M)$ into $\mathscr{B}(L)$.

Image and preimage $\mathscr{B}$-preserving maps $\quad f\colon L \to M$

image $\mathscr{B}$-preserving: $f[-]$ maps elements of $\mathscr{B}(L)$ into $\mathscr{B}(M)$.

preimage $\mathscr{B}$-preserving: $f_{-1}[-]$ maps elements of $\mathscr{B}(M)$ into $\mathscr{B}(L)$.

$\mathscr{B}$	**image $\mathscr{B}$-preserving**	**preimage $\mathscr{B}$-preserving**
$\mathfrak{c}$	closed maps	all

Image and preimage $\mathscr{B}$-preserving maps $\quad f\colon L \to M$

image $\mathscr{B}$-preserving: $f[-]$ maps elements of $\mathscr{B}(L)$ into $\mathscr{B}(M)$.

preimage $\mathscr{B}$-preserving: $f_{-1}[-]$ maps elements of $\mathscr{B}(M)$ into $\mathscr{B}(L)$.

$\mathscr{B}$	**image $\mathscr{B}$-preserving**	**preimage $\mathscr{B}$-preserving**
$\mathfrak{c}$	closed maps	all
$\mathfrak{o}$	open maps	all

Image and preimage $\mathscr{B}$-preserving maps $\quad f\colon L \to M$

image $\mathscr{B}$-preserving: $f[-]$ maps elements of $\mathscr{B}(L)$ into $\mathscr{B}(M)$.

preimage $\mathscr{B}$-preserving: $f_{-1}[-]$ maps elements of $\mathscr{B}(M)$ into $\mathscr{B}(L)$.

$\mathscr{B}$	**image $\mathscr{B}$-preserving**	**preimage $\mathscr{B}$-preserving**
$\mathfrak{c}$	closed maps	all
$\mathfrak{o}$	open maps	all
$\mathfrak{c}^*$	$f(a \vee f^*(b)) = f(a) \vee b$ regular	f^* of type E (e.g. nearly open) (Banaschewski & Pultr)

Image and preimage $\mathscr{B}$-preserving maps $\quad f\colon L \to M$

image $\mathscr{B}$-preserving: $f[-]$ maps elements of $\mathscr{B}(L)$ into $\mathscr{B}(M)$.

preimage $\mathscr{B}$-preserving: $f_{-1}[-]$ maps elements of $\mathscr{B}(M)$ into $\mathscr{B}(L)$.

$\mathscr{B}$	**image $\mathscr{B}$-preserving**	**preimage $\mathscr{B}$-preserving**
$\mathfrak{c}$	closed maps	all
$\mathfrak{o}$	open maps	all
$\mathfrak{c}^*$	$f(\underset{\sim}{a} \vee f^*(b)) = f(a) \vee b$ regular	f^* of type E (e.g. nearly open) (Banaschewski & Pultr)
$\mathfrak{c}_{\mathrm{coz}}$	$f(\underset{\sim}{a} \vee f^*(b)) = f(a) \vee b$ cozero	all

AIM II: to get DUAL results for free

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathrm{c}} : L \mapsto (\mathscr{B}(L))^{\mathrm{c}}$

$\mathscr{B}$ -normal:

$$\boxed{\mathfrak{c}(a) \vee \mathfrak{c}(b) = 1 \ \Rightarrow \ \exists u, v\colon \ \mathfrak{c}(u) \wedge \mathfrak{c}(v) = 1,\ \mathfrak{c}(a) \vee \mathfrak{c}(u) = 1 = \mathfrak{c}(b) \vee \mathfrak{c}(v)}$$

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathsf{c}} \colon L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}^{\mathsf{c}}$-normal: $\mathscr{B}$-disconnected.

$$\boxed{\mathfrak{o}(a) \vee \mathfrak{o}(b) = 1 \ \Rightarrow\ \exists u, v\colon\ \mathfrak{o}(u) \wedge \mathfrak{o}(v) = 1,\ \mathfrak{o}(a) \vee \mathfrak{o}(u) = 1 = \mathfrak{o}(b) \vee \mathfrak{o}(v)}$$

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathrm{c}}: L \mapsto (\mathscr{B}(L))^{\mathrm{c}}$

$\mathscr{B}^{\mathrm{c}}$-normal: $\mathscr{B}$-disconnected.

$$\boxed{\mathfrak{o}(a) \vee \mathfrak{o}(b) = 1 \;\Rightarrow\; \exists u, v\colon\; \mathfrak{o}(u) \wedge \mathfrak{o}(v) = 1,\; \mathfrak{o}(a) \vee \mathfrak{o}(u) = 1 = \mathfrak{o}(b) \vee \mathfrak{o}(v)}$$

$$\equiv [\mathfrak{c}(a) \wedge \mathfrak{c}(b) = 0 \;\Rightarrow\; \exists u, v\colon\; \mathfrak{c}(u) \vee \mathfrak{c}(v) = 0,\; \mathfrak{c}(a) \wedge \mathfrak{c}(u) = 0 = \mathfrak{c}(b) \wedge \mathfrak{c}(v)]$$

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathsf{c}}: L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}^{\mathsf{c}}$-normal: $\mathscr{B}$-disconnected.

$$\boxed{\mathfrak{o}(a) \vee \mathfrak{o}(b) = 1 \;\Rightarrow\; \exists u, v\colon\; \mathfrak{o}(u) \wedge \mathfrak{o}(v) = 1,\ \mathfrak{o}(a) \vee \mathfrak{o}(u) = 1 = \mathfrak{o}(b) \vee \mathfrak{o}(v)}$$

$$\equiv [\mathfrak{c}(a) \wedge \mathfrak{c}(b) = 0 \;\Rightarrow\; \exists u, v\colon\; \mathfrak{c}(u) \vee \mathfrak{c}(v) = 0,\ \mathfrak{c}(a) \wedge \mathfrak{c}(u) = 0 = \mathfrak{c}(b) \wedge \mathfrak{c}(v)]$$

$$\equiv [a \wedge b = 0 \;\Rightarrow\; \exists u, v \in L\colon\; u \vee v = 1,\ a \wedge u = 0 = b \wedge v]$$

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathsf{c}}: L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}^{\mathsf{c}}$-normal: $\mathscr{B}$-disconnected.

$$\boxed{\mathfrak{o}(a) \vee \mathfrak{o}(b) = 1 \ \Rightarrow\ \exists u, v\colon\ \mathfrak{o}(u) \wedge \mathfrak{o}(v) = 1,\ \mathfrak{o}(a) \vee \mathfrak{o}(u) = 1 = \mathfrak{o}(b) \vee \mathfrak{o}(v)}$$

$$\equiv [\mathfrak{c}(a) \wedge \mathfrak{c}(b) = 0 \ \Rightarrow\ \exists u, v\colon\ \mathfrak{c}(u) \vee \mathfrak{c}(v) = 0,\ \mathfrak{c}(a) \wedge \mathfrak{c}(u) = 0 = \mathfrak{c}(b) \wedge \mathfrak{c}(v)]$$

$$\equiv [a \wedge b = 0 \ \Rightarrow\ \exists u, v \in L\colon\ u \vee v = 1,\ a \wedge u = 0 = b \wedge v]$$

need only for a, b <u>regular</u> $\quad (a \wedge b = 0 \Leftrightarrow a^{**} \wedge b^{**} = 0)$

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathsf{c}} \colon L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}^{\mathsf{c}}$-normal: $\mathscr{B}$-disconnected.

$$\boxed{\mathfrak{o}(a) \vee \mathfrak{o}(b) = 1 \;\Rightarrow\; \exists u, v\colon\; \mathfrak{o}(u) \wedge \mathfrak{o}(v) = 1,\; \mathfrak{o}(a) \vee \mathfrak{o}(u) = 1 = \mathfrak{o}(b) \vee \mathfrak{o}(v)}$$

$$\equiv [\mathfrak{c}(a) \wedge \mathfrak{c}(b) = 0 \;\Rightarrow\; \exists u, v\colon\; \mathfrak{c}(u) \vee \mathfrak{c}(v) = 0,\; \mathfrak{c}(a) \wedge \mathfrak{c}(u) = 0 = \mathfrak{c}(b) \wedge \mathfrak{c}(v)]$$

$$\equiv [a \wedge b = 0 \;\Rightarrow\; \exists u, v \in L\colon\; u \vee v = 1,\; a \wedge u = 0 = b \wedge v]$$

need only for a, b regular $\qquad (a \wedge b = 0 \Leftrightarrow a^{**} \wedge b^{**} = 0)$

$$\equiv (a \wedge b)^* = a^* \vee b^*$$

(De Morgan frames)

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathsf{c}}: L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}^{\mathsf{c}}$-normal: $\mathscr{B}$-disconnected.

Selection $\mathscr{B}$	$\mathscr{B}$-normal frames	$\mathscr{B}$-disconnected frames
$\mathfrak{c}$	normal	extremally disconnected

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathrm{c}}\colon L \mapsto (\mathscr{B}(L))^{\mathrm{c}}$

$\mathscr{B}^{\mathrm{c}}$-normal: $\mathscr{B}$-disconnected.

Selection $\mathscr{B}$	**$\mathscr{B}$-normal frames**	**$\mathscr{B}$-disconnected frames**
$\mathfrak{c}$	normal	extremally disconnected
$\mathfrak{c}^*$	mildly normal	extremally disconnected

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathsf{c}}: L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}^{\mathsf{c}}$-normal: $\mathscr{B}$-disconnected.

Selection $\mathscr{B}$	$\mathscr{B}$-normal frames	$\mathscr{B}$-disconnected frames
$\mathfrak{c}$	normal	extremally disconnected
$\mathfrak{c}^*$	mildly normal	extremally disconnected
$\mathfrak{c}_\delta$	δ-normal	extremally δ-disconnected

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathsf{c}}\colon L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

$\mathscr{B}^{\mathsf{c}}$-normal: $\mathscr{B}$-disconnected.

Selection $\mathscr{B}$	**$\mathscr{B}$-normal frames**	**$\mathscr{B}$-disconnected frames**
$\mathfrak{c}$	normal	extremally disconnected
$\mathfrak{c}^*$	mildly normal	extremally disconnected
$\mathfrak{c}_\delta$	δ-normal	extremally δ-disconnected
$\mathfrak{c}_{\mathrm{coz}}$	all frames	F-frames

F-frame $\equiv$ every $\mathfrak{o}(\mathrm{coz}\, f)$ is C^*-embedded.

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathsf{c}}\colon L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving.

If L is $\mathscr{B}$-normal then M is also $\mathscr{B}$-normal.

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathrm{c}} \colon L \mapsto (\mathscr{B}(L))^{\mathrm{c}}$

Theorem

Let $f \colon L \to M$ be a surjective localic map such that

f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving.

If L is $\mathscr{B}$-normal then M is also $\mathscr{B}$-normal.

Just APPLY it to $\mathscr{B}^{\mathrm{c}}$!

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathrm{c}}\colon L \mapsto (\mathscr{B}(L))^{\mathrm{c}}$

Corollary

Let $f\colon L \to M$ be a surjective localic map such that

f is image $\mathscr{B}^{\mathrm{c}}$-preserving and preimage $\mathscr{B}$-preserving.

If L is $\mathscr{B}$-~~normal~~ then M is also $\mathscr{B}$-~~normal~~.

disconnected disconnected

preimage $\mathscr{B}^{\mathrm{c}}$-preserving = preimage $\mathscr{B}$-preserving

(because $f_{-1}[-]$ preserves complements)

ANOTHER FEATURE: dualization

$\mathscr{B}^{\mathsf{c}} \colon L \mapsto (\mathscr{B}(L))^{\mathsf{c}}$

Corollary

Let $f \colon L \to M$ be a surjective localic map such that

f is image $\mathscr{B}^{\mathsf{c}}$-preserving and preimage $\mathscr{B}$-preserving.

If L is $\mathscr{B}$-~~normal~~ disconnected then M is also $\mathscr{B}$-~~normal~~ disconnected.

preimage $\mathscr{B}^{\mathsf{c}}$-preserving = preimage $\mathscr{B}$-preserving

(because $f_{-1}[-]$ preserves complements)

Example $\mathscr{B} = \mathfrak{c}$:

Extremally disconnected locales are invariant under OPEN mappings.

ANOTHER FEATURE: dualization

localic map $f\colon L \to M$

image $\mathscr{B}^c$-preserving

preimage $\mathscr{B}^c$-preserving $\equiv$ preimage $\mathscr{B}$-preserving

$\mathscr{B}$	**image $\mathscr{B}^c$-preserving**	**preimage $\mathscr{B}^c$-preserving**
$\mathfrak{c}$	open	all

ANOTHER FEATURE: dualization

localic map $f\colon L \to M$

image $\mathscr{B}^{\mathsf{c}}$-preserving

preimage $\mathscr{B}^{\mathsf{c}}$-preserving $\equiv$ preimage $\mathscr{B}$-preserving

$\mathscr{B}$	**image $\mathscr{B}^{\mathsf{c}}$-preserving**	**preimage $\mathscr{B}^{\mathsf{c}}$-preserving**
$\mathfrak{c}$	open	all
$\mathfrak{c}^*$	nearly open	f^* of type E (e.g. nearly open) (Banaschewski & Pultr)

ANOTHER FEATURE: dualization

localic map $f\colon L \to M$

image $\mathscr{B}^{\mathsf{c}}$-preserving

preimage $\mathscr{B}^{\mathsf{c}}$-preserving $\equiv$ preimage $\mathscr{B}$-preserving

$\mathscr{B}$	**image $\mathscr{B}^{\mathsf{c}}$-preserving**	**preimage $\mathscr{B}^{\mathsf{c}}$-preserving**
$\mathfrak{c}$	open	all
$\mathfrak{c}^*$	nearly open	f^* of type E (e.g. nearly open) (Banaschewski & Pultr)
$\mathfrak{c}_{\mathrm{coz}}$	?	all

The perfect case

In spaces (Michael 1956):

$$\forall\, U \in \mathcal{O}(X)\ \exists\, (U_n)_{\mathbb{N}} \subseteq \mathcal{O}(X)\colon U = \bigcup_n U_n \ \text{ and }\ \overline{U_n} \subseteq U\ \forall n.$$

The perfect case

Perfect normality

In spaces (Michael 1956):

$$\forall\, U \in \mathcal{O}(X)\ \exists\ (U_n)_{\mathbb{N}} \subseteq \mathcal{O}(X)\colon U = \bigcup_n U_n \ \text{ and }\ \overline{U_n} \subseteq U\ \forall n.$$

In frames (Charalambous 1974):

$$\forall a \in L\ \ \exists (a_n)_{\mathbb{N}} \subseteq L\colon\ a = \bigvee_n a_n \ \text{ and }\ a_n \prec a\ \forall n.$$

The perfect case

In spaces (Michael 1956):

$$\forall\, U \in \mathcal{O}(X)\ \exists\ (U_n)_{\mathbb{N}} \subseteq \mathcal{O}(X):\ U = \bigcup_n U_n \ \text{ and } \ \overline{U_n} \subseteq U\ \forall n.$$

In frames (Charalambous 1974):

$$\forall a \in L\ \ \exists (a_n)_{\mathbb{N}} \subseteq L:\ \ a = \bigvee_n a_n \ \text{ and } \ a_n \prec a\ \forall n.$$

$$\Updownarrow$$

normal $+$ perfect

The perfect case

In spaces (Michael 1956):

$$\forall\, U \in \mathcal{O}(X)\ \exists\, (U_n)_{\mathbb{N}} \subseteq \mathcal{O}(X)\colon U = \bigcup_n U_n \ \text{ and } \ \overline{U_n} \subseteq U \ \forall n.$$

In frames (Charalambous 1974):

$$\forall a \in L\ \ \exists (a_n)_{\mathbb{N}} \subseteq L\colon\ a = \bigvee_n a_n \ \text{ and } \ a_n \prec a \ \forall n.$$

$$\Updownarrow$$

normal $+$ perfect every open is an F_σ-sublocale:

$$L \text{ is perfect} \equiv \forall\, \mathfrak{o}(a),\ \ \mathfrak{o}(a) = \bigwedge_{n\in\mathbb{N}} \mathfrak{c}(a_n)$$

The perfect case

Perfect normality

In spaces (Michael 1956):

$$\forall\, U \in \mathcal{O}(X)\ \exists\, (U_n)_{\mathbb{N}} \subseteq \mathcal{O}(X) \colon U = \bigcup_n U_n \ \text{ and } \ \overline{U_n} \subseteq U \ \forall n.$$

In frames (Charalambous 1974):

$$\forall a \in L \ \ \exists (a_n)_{\mathbb{N}} \subseteq L \colon \ a = \bigvee_n a_n \ \text{ and } \ a_n \prec a \ \forall n.$$

$$\Updownarrow$$

normal + perfect every open is an F_σ-sublocale:

$$L \text{ is perfect} \equiv \ \forall\, \mathfrak{o}(a), \ \ \mathfrak{o}(a) = \bigwedge_{n\in\mathbb{N}} \mathfrak{c}(a_n)$$

$$\boxed{L \text{ is } \mathscr{B}\text{-perfect} \equiv \ \forall\, A \in \mathscr{B}^{\mathsf{c}}, \ \ A = \bigwedge_{n\in\mathbb{N}} A_n \ \text{ with each } A_n \in \mathscr{B}}$$

The perfect case

$\mathscr{B}$	$\mathscr{B}$-perfect	$\mathscr{B}$-perfectly normal	$\mathscr{B}^{\mathrm{c}}$-perfect	$\mathscr{B}^{\mathrm{c}}$-perfectly normal
$\mathfrak{c}$				
$\mathfrak{c}^*$				
$\mathfrak{c}_{\mathrm{coz}}$				

The perfect case

$\mathscr{B}$	$\mathscr{B}$-perfect	$\mathscr{B}$-perfectly normal	$\mathscr{B}^{\mathrm{c}}$-perfect	$\mathscr{B}^{\mathrm{c}}$-perfectly normal
$\mathfrak{c}$	perfect	perfectly normal	Boolean	Boolean
$\mathfrak{c}^*$				
$\mathfrak{c}_{\mathrm{coz}}$				

The perfect case

$\mathscr{B}$	$\mathscr{B}$-perfect	$\mathscr{B}$-perfectly normal	$\mathscr{B}^{\mathrm{c}}$-perfect	$\mathscr{B}^{\mathrm{c}}$-perfectly normal
$\mathfrak{c}$	perfect	perfectly normal	Boolean	Boolean
$\mathfrak{c}^*$	?	OZ frames	?	extremally disconn.
$\mathfrak{c}_{\mathrm{coz}}$				

OZ frame $\equiv$ every regular element is a cozero.

The perfect case

$\mathscr{B}$	$\mathscr{B}$-perfect	$\mathscr{B}$-perfectly normal	$\mathscr{B}^{\mathrm{c}}$-perfect	$\mathscr{B}^{\mathrm{c}}$-perfectly normal
$\mathfrak{c}$	perfect	perfectly normal	Boolean	Boolean
$\mathfrak{c}^*$	?	OZ frames	?	extremally disconn.
$\mathfrak{c}_{\mathrm{coz}}$	?	all frames	?	*P*-frames

OZ frame $\equiv$ every regular element is a cozero.

P-frame $\equiv$ $\mathrm{Coz}\, L$ is complemented.

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f f_{-1}[B] = B$ for every $B \in \mathscr{B}^{\mathsf{c}}(M)$.

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f f_{-1}[B] = B$ for every $B \in \mathscr{B}^{\mathsf{c}}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f f_{-1}[B] = B$ for every $B \in \mathscr{B}^{c}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

Proof:

$$L \xrightarrow{f} M$$

The perfect case

Theorem

Let $f: L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f f_{-1}[B] = B$ for every $B \in \mathscr{B}^{\mathsf{c}}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

Proof:

$$L \xrightarrow{\quad f \quad} M$$

$$B \in \mathscr{B}^{\mathsf{c}}(M)$$

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f f_{-1}[B] = B$ for every $B \in \mathscr{B}^{\mathrm{c}}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

Proof:

$$L \xrightarrow{\quad f \quad} M$$

$$f f_{-1}[B] = B \in \mathscr{B}^{\mathrm{c}}(M)$$

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f\, f_{-1}[B] = B$ for every $B \in \mathscr{B}^{\mathsf{c}}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

Proof:

$$L \xrightarrow{\ f\ } M$$

$$f\, \underbrace{f_{-1}[B]}_{\in\, \mathscr{B}^{\mathsf{c}}(L)} = B \in \mathscr{B}^{\mathsf{c}}(M)$$

$f_{-1}[-]$ preserves complements

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f f_{-1}[B] = B$ for every $B \in \mathscr{B}^{\mathsf{c}}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

Proof:

$$L \xrightarrow{f} M$$

$$f[\textstyle\bigwedge_n A_n] \quad = \quad f f_{-1}[B] = B \in \mathscr{B}^{\mathsf{c}}(M)$$

$A_n \in \mathscr{B}(L)$ L is $\mathscr{B}$-perfect $\in \mathscr{B}^{\mathsf{c}}(L)$

$f_{-1}[-]$ preserves complements

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f\, f_{-1}[B] = B$ for every $B \in \mathscr{B}^{\mathsf{c}}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

Proof:

$$L \xrightarrow{\quad f \quad} M$$

$$\underbrace{\bigwedge_n f[A_n]}_{f[-] \text{ is a localic map}} = f[\bigwedge_n A_n] = \underbrace{f\, f_{-1}[B]}_{\in \mathscr{B}^{\mathsf{c}}(L)} = B \in \mathscr{B}^{\mathsf{c}}(M)$$

$A_n \in \mathscr{B}(L)$

L is $\mathscr{B}$-perfect

$f_{-1}[-]$ preserves complements

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f\, f_{-1}[B] = B$ for every $B \in \mathscr{B}^{\mathsf{c}}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

Proof:

$$L \xrightarrow{\quad f \quad} M$$

$$\underbrace{\textstyle\bigwedge_n f[A_n]}_{f[-] \text{ is a localic map}} = f[\textstyle\bigwedge_n A_n] \underset{L \text{ is } \mathscr{B}\text{-perfect}}{=} f\, \underbrace{f_{-1}[B]}_{\in \mathscr{B}^{\mathsf{c}}(L)} = B \in \mathscr{B}^{\mathsf{c}}(M)$$

$f_{-1}[-]$ preserves complements

$$A_n \in \mathscr{B}(L) \;\mapsto\; f[A_n] \in \mathscr{B}(M).$$

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f\, f_{-1}[B] = B$ for every $B \in \mathscr{B}^{c}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

Example $\mathscr{B} = \mathfrak{c}$:

Perfect locales are invariant under CLOSED mappings.

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f\, f_{-1}[B] = B$ for every $B \in \mathscr{B}^{\mathrm{c}}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

Example $\mathscr{B} = \mathfrak{c}$:

Perfect locales are invariant under CLOSED mappings.

Perfectly normal locales are invariant under CLOSED mappings.

The perfect case

Theorem

Let $f\colon L \to M$ be a surjective localic map such that

- f is image $\mathscr{B}$-preserving and preimage $\mathscr{B}$-preserving
- $f f_{-1}[B] = B$ for every $B \in \mathscr{B}^{\mathrm{c}}(M)$.

If L is $\mathscr{B}$-perfect then M is also $\mathscr{B}$-perfect.

Example $\mathscr{B} = \mathfrak{c}$:

Perfect locales are invariant under CLOSED mappings.

Perfectly normal locales are invariant under CLOSED mappings.

Boolean locales are invariant under OPEN mappings.

Other interesting cases

Hereditary case:

hereditary normality: every its sublocale is $\mathscr{B}$-normal.

(suffices for every sublocale in $\mathscr{B}^{\mathsf{c}}$)

Other interesting cases

Hereditary case:

hereditary normality: every its sublocale is $\mathscr{B}$-normal.

$\Updownarrow$ (suffices for every sublocale in $\mathscr{B}^{\mathsf{c}}$)

complete normality

MORE

Other interesting cases

Hereditary case:

hereditary normality: every its sublocale is $\mathscr{B}$-normal.

$\Updownarrow$ (suffices for every sublocale in $\mathscr{B}^c$)

complete normality

MORE

Real functions:

$\mathscr{B}$-continuity, $\mathscr{B}$-semicontinuity, general insertion theorems...

Main references

- J. Gutiérrez García and J. Picado,
  On the parallel between normality and extremal disconnectedness,
  J. Pure Appl. Algebra 218 (2014) 784-803.

- J. Gutiérrez García, T. Kubiak and J. Picado,
  Perfectness in locales,
  Quaest. Math., in press.

- J. Gutiérrez García, T. Kubiak and J. Picado,
  On extremal disconnectedness and its hereditary property,
  in preparation.

- J. Picado and A. Pultr,
  Frames and Locales: topology without points,
  Frontiers in Mathematics, Vol. 28, Springer, Basel, 2012.