Pervin spaces

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Outline

(1) Motivations
(2) Pervin spaces
(3) Complete Pervin spaces
(4) Completion of a Pervin space
(5) Quotient maps
(6) Duality results
(7) Back to languages

Disclaimer. Most of the results of this lecture are standard, only the perspective might be new.

References

Inspiring articles include:

- Several articles by J. ALMEIDA and by A. COSTA
- Á. CSÁSZÁR, D-completions of Pervin-type quasi-uniformities (1993)
- M. ERNÉ, Ideal Completions and Compactifications (2001)
- H.-P. A. KÜNZI, An introduction to quasi-uniform spaces (2009)
- W. J. PERVIN, Quasi-uniformization of topological spaces (1962)

Part I

Motivations

Dual spaces in language theory

The set of finite or infinite words on A is

- The dual space of the Boolean algebra generated by the languages of the form uA^* where u is a word.
- The completion of the metric space A^*, equipped with the prefix metric.

The free profinite monoid on A is

- The dual space of the Boolean algebra of regular languages.
- The completion of the metric space A^*, equipped with the profinite metric.

Dual spaces in language theory

The space $A^* \cup \{\infty\}$ is

- The dual space of the Boolean algebra of finite or cofinite languages.
- The completion of the metric space A^*, equipped with $d(u, v) = 2^{-\min\{|u|,|v|\}}$.

The Stone-Čech compactification βA^* of A^* is

- The dual space of the Boolean algebra of all languages.
- The completion of the uniform space A^*, equipped with the discrete uniform structure.

Equational description of languages

The Boolean algebra generated by the languages of the form uA^* can be defined by the profinite equations $x^\omega y = x^\omega$.

Finite-cofinite languages can be defined by the profinite equations $x^\omega y = x^\omega = yx^\omega$.

and many more...

$\triangleright$ In the profinite monoid, x^ω is defined as the limit of the Cauchy sequence $x^{n!}$.

Equations

Theorem (GGP 08)

Every Boolean algebra of regular languages of A^* *can be defined by a set of equations of the form* $u = v$*, where* u, v *are profinite words.*

Theorem (GGP 10)

Every Boolean algebra of A^* *can be defined by a set of equations of the form* $u = v$*, where* $u, v \in \beta A^*$*.*

Motivations

Is it possible to extend these results to concrete Boolean algebras? More precisely:

- Can one obtain the dual of a Boolean algebra as a completion of some sort?
- What about distributive lattices and Stone-Priestley duality?
- Is there a way to extend the equational approach to the lattice case?

Motivations

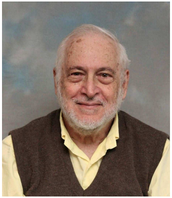
Is it possible to extend these results to concrete Boolean algebras? More precisely:

- Can one obtain the dual of a Boolean algebra as a completion of some sort?
- What about distributive lattices and Stone-Priestley duality?
- Is there a way to extend the equational approach to the lattice case?

Answer: yes, use the completion of a Pervin space!

Part II

Pervin spaces



A Pervin space is a pair $(X, \mathcal{L})$ where $\mathcal{L}$ is a lattice of subsets of X.

The elements of $\mathcal{L}$ are called the blocks of the Pervin space.

A morphism of Pervin spaces is a map between two Pervin spaces such that the inverse image of each block is a block.

Examples of Pervin spaces

Examples on $X = \{0, 1\}$

- Boolean space $\mathbb{B}$: $\mathcal{L} = \Big\{\emptyset, \{0\}, \{1\}, \{0, 1\}\Big\}$
- Sierpiński space $\mathbb{S}$: $\mathcal{L} = \Big\{\emptyset, \{1\}, \{0, 1\}\Big\}$.

Examples on $X = \mathbb{N}$

- $\mathcal{L} = \{\emptyset\} \cup \{\text{cofinite subsets of } \mathbb{N}\}$.
- $\mathcal{L} = \{\mathbb{N}\} \cup \{\text{finite subsets of } \mathbb{N}\}$.
- $\mathcal{L} = \{\text{finite/cofinite subsets of } \mathbb{N}\}$.

The three faces of a Pervin space

- Partially ordered set
- Topological space
- Quasi-uniform space

The three faces

Partially ordered set. $x \leqslant y$ iff, for each block L,

$$x \in L \Rightarrow y \in L$$

Topological space. The blocks of a Pervin space form a basis of its topology. Therefore, the open sets are the (possibly infinite) union of blocks.

Quasi-uniform space. The sets

$$U_L = \{(x, y) \in X \times X \mid x \in L \Rightarrow y \in L\}$$

form a subbasis of its quasi-uniformity.

Quasi-uniform spaces

A quasi-uniformity on a set X is a nonempty set $\mathcal{U}$ of subsets of $X \times X$ (the entourages) satisfying the following properties:

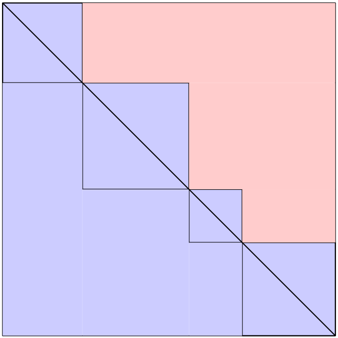
(1) Every subset of $X \times X$ containing an entourage is an entourage,

(2) the intersection of any two entourages contains an entourage,

(3) each entourage contains the diagonal of $X \times X$,

(4) for each entourage U, there exists an entourage V such that $V \circ V \subseteq U$.

The entourages of a Pervin space



Pervin spaces among quasi-uniform spaces

Theorem

A quasi-uniform space is a Pervin space iff it is transitive and totally bounded.

- A quasi-uniformity is transitive if it has a basis consisting of transitive entourages.
- A quasi-uniform space X is totally bounded if, for each entourage U, there exists a finite cover $\mathcal{C}$ of X such that $C \times C \subseteq U$ for each $C \in \mathcal{C}$.

The category of Pervin spaces

Definition. A function $f : X \to Y$ is uniformly continuous if, for each block L of Y, $f^{-1}(L)$ is a block of X.

Any uniformly continuous map is continuous and increasing, but the converse is not true.

Pervin spaces together with uniformly continuous maps form the category $\mathbf{Perv}$ of Pervin spaces.

Most standard categorical notions extend smoothly from topological spaces to Pervin spaces (products, coproducts, etc.).

Preorders on a Pervin space $(X, \mathcal{L})$

Preorder: $x \leqslant y$ iff, for each block L,

$$x \in L \Rightarrow y \in L$$

Specialisation preorder: $x \leqslant_s y$ if and only y belongs to every open set that contains x.

Uniform preorder: $x \leqslant_{\mathcal{U}} y$ iff (x, y) belongs to all the entourages.

In a Pervin space, these three preorders coincide.

Equivalence relation: $x \sim y$ iff, for each block L,

$$x \in L \Leftrightarrow y \in L$$

The Kolmogorov quotient of a Pervin space $(X, \mathcal{L})$

Proposition (trivial)

The following conditions are equivalent:

(1) $\leqslant$ *is an order,*

(2) $\sim$ *is the equality relation,*

(3) X *is a Kolmogorov (T_0) space.*

The quotient space $(X/{\sim}, \mathcal{L}/{\sim})$ is the Kolmogorov quotient of the Pervin space $(X, \mathcal{L})$.

Metrizability

Proposition

Let $\mathcal{L}$ be a Boolean algebra of subsets of X. Are equivalent:

(1) *The associated Pervin space is metrizable,*
(2) *The uniformity has a countable basis,*
(3) *$\mathcal{L}$ is countable.*

Similar results for lattices/hemi-metrizability.
Warning! Having a countable basis for the topology does not suffice.

Part III

Complete Pervin spaces

The symmetrical topology

Let $(X, \mathcal{L})$ be a Pervin space. Let $\mathcal{L}^s$ be the Boolean algebra generated by $\mathcal{L}$ and let

$$D(\mathcal{L}) = \{L_1 - L_0 \mid L_0, L_1 \in \mathcal{L}\}.$$

Proposition (Hausdorff)

The Boolean algebra $\mathcal{L}^s$ consists of the finite unions of elements of $D(\mathcal{L})$.

Then $(X, \mathcal{L}^s)$ is a Pervin space. Its topology is the symmetrical topology on X defined by $\mathcal{L}$.

Cauchy filters and complete Pervin spaces

Let $(X, \mathcal{L})$ be a Pervin space.

Definition. A filter $\mathcal{F}$ on X is Cauchy if, for every $L \in \mathcal{L}$, either $L \in \mathcal{F}$ or $L^c \in \mathcal{F}$.

Fact. A cluster point of a Cauchy filter is a limit point.

Definition. A Pervin space is complete if every Cauchy filter converges in the symmetrical topology.

Properties of complete spaces

Theorem

Let $(X, \mathcal{L})$ be a Pervin space. Are equivalent:

(1) $(X, \mathcal{L})$ *is complete,*

(2) $(X, \mathcal{L}^s)$ *is complete,*

(3) $(X, \mathcal{L}^s)$ *is compact.*

If these conditions are satisfied, then $(X, \mathcal{L})$ is compact.

However a compact Pervin space needs not to be complete: take $\mathbb{N}$ with the lattice of cofinite sets.

Specifications for the completion of a Pervin space

A completion of a Pervin space $(X, \mathcal{L})$ is a complete Kolmogorov Pervin space $(\widehat{X}, \widehat{\mathcal{L}})$ together with a uniformly continuous map $\imath \colon X \to \widehat{X}$ satisfying the following universal property:

for each uniformly continuous map $\varphi : X \to Y$, where Y is a complete Kolmogorov Pervin space, there exists a unique uniformly continuous map $\widehat{\varphi} : (\widehat{X}, \widehat{\mathcal{L}}) \to (Y, \mathcal{V})$ and such that $\widehat{\varphi} \circ \imath = \varphi$.

This implies unicity (up to isomorphism), but not existence!

Part IV

Building the completion

Valuations on a lattice of subsets $\mathcal{L}$

A valuation is a lattice morphism from $\mathcal{L}$ to the Boolean algebra $\{0, 1\}$.

Thus a map $v : \mathcal{L} \to \{0, 1\}$ is a valuation if, for all $L_1, L_2 \in \mathcal{L}$,

(1) $v(\emptyset) = 0$, $v(X) = 1$,

(2) $v(L_1 \cap L_2) = v(L_1)v(L_2)$,

(3) $v(L_1 \cup L_2) = v(L_1) + v(L_2)$.

where the sum and the product are the Boolean operations.

Valuations and prime filters are the same things

If $v : \mathcal{L} \to \{0, 1\}$ is a valuation, then the set $v^{-1}(1)$ is a prime filter.

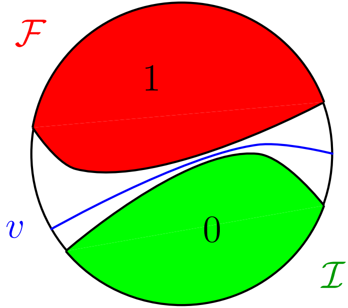
If $\mathcal{P}$ is a prime filter of $\mathcal{L}$, then its characteristic function

$$v(L) = \begin{cases} 1 & \text{if } L \in \mathcal{P} \\ 0 & \text{otherwise} \end{cases}$$

is a valuation.

All you need are separations results (M. Erné)

Theorem. Let $\mathcal{I}$ be an ideal of $\mathcal{L}$ and let $\mathcal{F}$ be a filter of $\mathcal{L}$ disjoint from $\mathcal{I}$. Then there is a valuation v on $\mathcal{L}$ such that $v = 1$ on $\mathcal{F}$ and $v = 0$ on $\mathcal{I}$.



Other separations results

Let $\mathcal{I}$ be an ideal of $\mathcal{L}$ and let L be an element of $\mathcal{L} - \mathcal{I}$. Then there is a valuation v on $\mathcal{L}$ such that $v(L) = 1$ and $v = 0$ on $\mathcal{I}$.

Let $\mathcal{K}$ be a sublattice of a lattice $\mathcal{L}$ and let v be a valuation on $\mathcal{K}$. Then there exists a valuation on $\mathcal{L}$ whose restriction to $\mathcal{K}$ coincides with v.

Let $\mathcal{K}$ be a sublattice of a lattice $\mathcal{L}$ and let L be an element of $\mathcal{L} - \mathcal{K}$. Then there exist two valuations v_0 and v_1 on $\mathcal{L}$ such that $v_0(L) = 0$, $v_1(L) = 1$ and $v_1 \leqslant v_0$ on $\mathcal{K}$. Moreover, if $\mathcal{K}$ is a Boolean algebra, then $v_0 = v_1$ on $\mathcal{K}$.

Completion of a Pervin space $(X, \mathcal{L})$

Let $\widehat{X}$ be the set of all valuations on $\mathcal{L}$.
For each $L \in \mathcal{L}$, let

$$\widehat{L} = \{v \text{ is a valuation such that } v(L) = 1\}$$

Theorem

The completion of a Pervin space $(X, \mathcal{L})$ is the Pervin space $(\widehat{X}, \widehat{\mathcal{L}})$, where $\widehat{\mathcal{L}}$ is the lattice of subsets of $\widehat{X}$ defined by $\widehat{\mathcal{L}} = \{\widehat{L} \mid L \in \mathcal{L}\}$.

Embedding $(X, \mathcal{L})$ into $(\widehat{X}, \widehat{\mathcal{L}})$

For each $x \in X$, let v_x be the valuation defined by

$$v_x(L) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{if } x \notin L \end{cases}$$

If $\leqslant$ is an order (that is, if $(X, \mathcal{L})$ is Kolmogorov), the map $x \to v_x$ defines an injective and uniformly continuous embedding from $(X, \mathcal{L})$ into $(\widehat{X}, \widehat{\mathcal{L}})$. Furthermore, X is dense in $\widehat{X}$.

Uniformly continuous extensions

Theorem

Every uniformly continuous map $f : X \to Y$ *admits a unique uniformly continuous extension* $\widehat{f} : \widehat{X} \to \widehat{Y}$, *given by* $\widehat{f}(v)(L) = v(f^{-1}(L))$.

Corollary

Let f_1 *and* f_2 *be two uniformly continuous maps from* X *to* Y. *If* $f_1 \leqslant f_2$, *then* $\widehat{f_1} \leqslant \widehat{f_2}$.

Towards duality

If $\mathcal{K}$ is a sublattice of $\mathcal{L}$, then the identity on X is a uniformly continuous map from $(X, \mathcal{L})$ to $(X, \mathcal{K})$.

Highly wanted property: the completion of the identity is a quotient map from $\widehat{(X, L)}$ onto $\widehat{(X, K)}$.

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Solution. Change the definition!

Part V

Quotient maps

Final maps

The solution is to consider the category $\mathbf{Perv}$ of Pervin spaces over the category of preordered sets.

Definition

A uniformly continuous map $\upsilon : X \to Y$ in $\mathbf{Perv}$ is final if, for each Pervin space Z, any increasing map $\psi : Y \to Z$ such that $\psi \circ \varphi$ is uniformly continuous is uniformly continuous.

A quotient map is a surjective final map.

Characterizations of final maps

Let $\varphi : X \to Y$ be a uniformly continuous map. Are equivalent:

(1) φ is a final map,

(2) for any upset L of Y such that $\varphi^{-1}(L)$ is a block of X, L is a block of Y.

(3) every increasing map ψ from Y to the Sierpiński space such that $\psi \circ \varphi$ is uniformly continuous is also uniformly continuous.

Example: The identity map is a quotient map from $\mathbb{B}$ onto $\mathbb{S}$.

Characterizations of final maps

Proposition

A uniformly continuous map is final iff it is a co-inequalizer of some pair of uniformly continuous maps.

Co-inequalizers

Let $Z \underset{g}{\overset{f}{\rightrightarrows}} Y$ be a pair of morphisms. A morphism $h : Y \to X$ is called a co-inequalizer of f and g if the following conditions hold:

(1) $h \circ f \leqslant h \circ g$,

(2) for every morphism $h' : Y \to X'$ such that $h' \circ f \leqslant h' \circ g$, there exists a unique morphism $\bar{h} : X \to X'$ such that $h' = \bar{h} \circ h$.

$$\begin{tikzcd} Z \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] \arrow[dr, "h'"'] & X \arrow[d, "\bar{h}"] \\ & & X' \end{tikzcd}$$

The highly wanted result

Theorem

If $\mathcal{K}$ is a sublattice of $\mathcal{L}$, then the identity on X is a uniformly continuous map from $(X, \mathcal{L})$ to $(X, \mathcal{K})$ and its completion is a quotient map from $\widehat{(X, L)}$ onto $\widehat{(X, K)}$.

Part VI

Duality results

Blocks and compact Pervin spaces

Theorem

Let $(X, \mathcal{L})$ be Kolmogorov Pervin space that is compact for the symmetrical topology and let L be a subset of X. Then the following conditions are equivalent:

(1) *L is a block of $\mathcal{L}$,*

(2) *L is compact open in $(X, \mathcal{L})$,*

(3) *L is an upset in $(X, \mathcal{L})$ and is clopen in $(X, \mathcal{L}^s)$,*

(4) *L is an upset in $(X, \mathcal{L})$ and a block in $(X, \mathcal{L}^s)$.*

Compact Pervin spaces

Theorem

Let $(X, \mathcal{L})$ be a Kolmogorov Pervin space. If $(X, \mathcal{L}^s)$ is compact, then $(X, \mathcal{L})$ is spectral.

A partial preorder is directed-complete if each of its directed subsets has a supremum.

Theorem

Let $(X, \mathcal{L})$ be a Kolmogorov Pervin space. If $(X, \mathcal{L}^s)$ is compact, then the preorder of $(X, \mathcal{L})$ is directed-complete.

Duality results

Theorem (Duality theorem)

The lattice $\widehat{\mathcal{L}}$ is the set of all compact open subsets of $\widehat{X}$. In particular, $\widehat{X}$ is compact.
Moreover, the maps $L \mapsto \widehat{L}$ and $K \mapsto K \cap X$ are mutually inverse lattice isomorphisms between $\mathcal{L}$ and $\widehat{\mathcal{L}}$.

Corollary

The completion of the Pervin space $(X, \mathcal{L})$ is equal to the Stone dual of $\mathcal{L}$.

Inequations

Let $(X, \mathcal{L})$ be a Pervin space, let L be a block of X and let (v, w) be a pair of valuations on $\mathcal{L}$. Then L satisfies the $\mathcal{L}$-inequation $v \leqslant w$ if $v(L) \leqslant w(L)$.

Given a set S of inequations, L satisfies S if it satisfies all the inequations of S.

A set of blocks $\mathcal{K}$ satisfies the inequation $v \leqslant w$ if $v(K) \leqslant w(K)$ for all $K \in \mathcal{K}$. Similarly, $\mathcal{K}$ satisfies S if it satisfies all the inequations of S.

Finally, the set of all blocks satisfying S is called the set of blocks defined by S.

Characterization of sublattices by inequations

Theorem

Let $(X, \mathcal{L})$ be a Pervin space. A set of blocks of X is a sublattice of $\mathcal{L}$ iff it can be defined by a set of $\mathcal{L}$-inequations.

Part VII

Back to languages

Inequations on languages

Theorem (GGP 08)

A set of regular languages of A^* is a lattice iff it can be defined by a set of inequations of the form $u \leqslant v$, where u, v are profinite words.

Theorem (GGP 10)

A set of languages of A^* is a lattice iff it can be defined by a set of equations of the form $u \leqslant v$, where $u, v \in \beta A^*$.

Syntactic preorder

The syntactic preorder of a language L of A^* is the relation $\leqslant_L$ defined on A^* by $u \leqslant_L v$ iff, for every $x, y \in A^*$, $xuy \in L \Rightarrow xvy \in L$.

Let $\mathcal{L}$ be the lattice generated by the quotients of a language L.

The preorder of the Pervin space $(A^*, \mathcal{L})$ is the syntactic preorder of L and $\sim$ is its syntactic congruence.