Principal quantales and effective equivalence relations

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Motivations

- Resende, 2007: Inverse quantal frames $\Longleftrightarrow$ étale groupoids:
  (This correspondence is not functorial in the usual sense. However it is in the bicategorical sense.)
- Protin and Resende, 2012: Open quantal frames $\Longleftrightarrow$ open groupoids.

In this talk

- A generalization of supported quantales which applies to non-unital quantales and to open groupoids.
- Q. and Resende, 2015: Groupoid quantales $\Longleftrightarrow$ open groupoids: in a way that makes feasible the study of actions and sheaves on such quantales (work in progress)
- Q. and Resende, 2015: Principal groupoid quantales $\Longleftrightarrow$ effective equivalence relations (étale-complete groupoids in a simplified situation).

Notation and terminology

- If X is a locale we refer to $\mathcal{O}(X)$ as itself in the dual category $Frm = Loc^{op}$ of frames.
- If $f : X \to Y$ is a map of locales we shall refer to the frame homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ that defines it as its inverse image. If f is an open map, the left adjoint to f^* is referred to as the direct image of f and it is denoted by $f_! : \mathcal{O}(X) \to \mathcal{O}(Y)$.
- The product of X and Y in **Loc** is $X \times Y$. it coincides with the tensor product in the category of sup-lattices of $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ (coproduct in **Frm**). Then, we write $\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes \mathcal{O}(Y)$.

Groupoids

$$
G = \quad \begin{tikzcd} G_2 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "r"] \arrow[r, shift right=2, "d"'] & G_0 \arrow[l, "u" description] \end{tikzcd}
$$

where G_2 is the pullback of the *domain* and *range* maps:

$$
\begin{tikzcd} G_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & G_1 \arrow[d, "r"] \\ G_1 \arrow[r, "d"'] & G_0 \end{tikzcd}
\qquad
\begin{tikzcd} G_2 \arrow[r, "\pi_1"] \arrow[d, "m"'] & G_1 \arrow[d, "d"] \\ G_1 \arrow[r, "d"'] & G_0 \end{tikzcd}
$$

d open $\Rightarrow$ m open (*open groupoids*)
d local homeomorphism $\Rightarrow$ m local homeomorphism (*étale groupoids*)

Effective equivalence relations

An *effective equivalence relation* is meant to be a open groupoid G which is the kernel pair of its co-equalizer. In other words, this means that the following square:

$$\begin{tikzcd} G_1 \arrow[r, "r"] \arrow[d, "d"'] & G_0 \arrow[d, two heads] \\ G_0 \arrow[r, two heads] & G_0/G \end{tikzcd} ,$$

where G_0/G is the *orbit locale*, is a pull-back in the category of *Loc*.

Since m is an open map, there is a sup-lattice homomorphism defined as:

$$\mathcal{O}(G_1) \otimes \mathcal{O}(G_1) \twoheadrightarrow \mathcal{O}(G_2) \xrightarrow{m_!} \mathcal{O}(G_1)$$

this defines an associative multiplication on $\mathcal{O}(G_1)$ which together with the isomorphism

$$\mathcal{O}(G_1) \xrightarrow{i_!} \mathcal{O}(G_1)$$

makes $\mathcal{O}(G_1)$ an involutive quantale. This quantale is denoted by $\mathcal{O}(G)$. It is a worth to mention that the involutive quantale $\mathcal{O}(G)$ of an open groupoid G is unital if and only if G is ètale, in which case the unit is $e = u_!(1)$.

Involutive quantales

A involutive quantale Q is an involutive semigroup,

- $(ab)c = a(bc)$
- $a^{**} = a$
- $(ab)^* = b^* a^*$
- $ae = a$ and $ea = a$ (unital quantale)

in the category of sup-lattices:

- $(\bigvee a_i)b = \bigvee a_i b$
- $b(\bigvee a_i) = \bigvee b a_i$
- $(\bigvee a_i)^* = \bigvee a_i^*$

Stable quantales

Let Q be a unital involutive quantale. We recall that by a *support* on Q is meant a sup-lattice homomorphism $\varsigma : Q \to Q$ satisfying the following conditions for all $a \in Q$:

$$\varsigma(a) \leq e$$
$$\varsigma(a) \leq aa^*$$
$$a \leq \varsigma(a)a.$$

The support is said to be *stable*, if in addition we have, for all $a, b \in Q$:

$$\varsigma(ab) \leq \varsigma(a).$$

We remark that the quantale $\mathcal{O}(G)$ of an étale groupoid G has a stable support given by $u_! \circ d_! : \mathcal{O}(G) \to \mathcal{O}(G)$.

Inverse quantal frames

Let Q be an inverse quantal frame, i.e., a stable quantal frame, satisfying

$$\bigvee Q_{\mathcal{I}} = 1_Q$$

where $Q_{\mathcal{I}} = \{a \in Q | a^* a \vee a^* a \leq e\}$ (partial units of Q)

Remark (Resende 2007)

Then $Q = \mathcal{O}(G)$ for an étale groupoid G

A-A-quantales

Definition

Let A be a locale. An A-A-bimodule M is a sup-lattice equipped with two unital (resp. left and right) A-module structures

$$(a, m) \mapsto {}_a|m \quad \text{and} \quad (a, m) \mapsto m|_a\,,$$

satisfying the following additional condition for all $a, b \in A$ and $m \in M$:

$$({}_a|m)|_b = {}_a|(m|_b)\,. \tag{1}$$

A morphism of A-A-bimodules is, of course, a sup-lattice homomorphism that preserves both actions.

The notation for the left and the right action is meant to convey the idea of restriction, as in the following example:

Example

Let A and M be locales, and $d, c : M \to A$ two continuous maps. Then the frame homomorphisms $d^*, c^* : A \to M$ make M an A-A-bimodule with the actiones defined by

$$
\begin{aligned}
{}_a|m &= d^*(a) \wedge m\,, \\
m|_a &= c^*(a) \wedge m\,.
\end{aligned}
$$

Definition

An A-A-quantale Q is just a semigroup in the monoidal category of A-A-bimodules, i.e. Q is an A-A-bimodule equipped with a quantale multiplication $(x, y) \mapsto xy$ *satisfying the following additional conditions for all* $a \in A$ *and* $x, y \in Q$*:*

$$({}_a|x)y = {}_a|(xy) \tag{2}$$

$$(x|_a)y = x({}_a|y) \tag{3}$$

$$(xy)|_a = x(y|_a) \tag{4}$$

Definition

An A-A-quantale is involutive if it is an involutive semigroup; the involution is denoted by $a \mapsto a^*$ and it is required to satisfy, besides the standard conditions $x^{**} = x$ and $(xy)^* = y^* x^*$, the following two conditions:

$$(\bigvee_i x_i)^* = \bigvee_i x_i^* \tag{5}$$

$$({}_a|x|_b)^* = {}_b|x^*|_a \tag{6}$$

It is worth remarking that an involutive A-A-quantale is not the same as an involutive semigroup in the category of A-A-bimodules.

General Supports

Definition

Let A be a locale. An involutive A-A-quantale Q is supported if it is equipped with a sup-lattice homomorphism $\varsigma : Q \to A$ satisfying the following conditions for all $x, y \in Q$:

1. $\varsigma(1_Q) = 1_A$.
2. ${}_{\varsigma(x)}|y \leq xx^*y$.
3. ${}_{\varsigma(x)}|x = x$.

A supported A-A-quantale (Q, ς) is an involutive A-A-quantale equipped with a specified support.

Let us denote by $R(Q)$ the set of *right-sided* elements of Q, an element $a \in Q$ is right-sided if $a1_Q \leq a$. Similarly, the set of *left-sided* elements of Q is denoted by $L(Q)$ and the set of *two-sided* elements is denoted by $T(Q)$.

Properties

Lemma

Let (Q, ς) be a supported A-A-quantale. The following properties can be derived from the axioms, for all $x, y \in Q$:

1. $({}_{\varsigma(x)}|y)^* = (y^*|_{\varsigma(x)})$.
2. $y|_{\varsigma(x)} \leq yxx^*$.
3. $x \leq xx^*x$.
4. $x \leq x1_Q$.
5. $1_Q 1_Q = 1_Q$.
6. ${}_{\varsigma(x1_Q)}|1_Q = x1_Q$.
7. *The sup-lattice homomorphism $\varsigma(Q) \to R(Q)$ defined by $x \mapsto {}_x|1_Q$ is a retraction split by the map $R(Q) \to \varsigma(Q)$ which is defined by $x \mapsto \varsigma(x)$; the map $\varsigma : R(Q) \to A$ is an embedding.*

Equivariant supports

Definition

A support ς is said to be equivariant if

$$\varsigma({}_a|x) = a \wedge \varsigma(x)\,. \tag{7}$$

An A-A-quantale (Q, ς) is called equivariantly supported whenever the support is equivariant.

Lemma

Let (Q, ς_Q) be an equivariantly supported A-A-quantale. Then, the map $A \to R(Q)$ defined by $x \mapsto {}_x|1_Q$ is order isomorphism whose inverse is the map $R(Q) \to A$ defined by $x \mapsto \varsigma(x)$. In particular $R(Q) \cong A$, i.e., $R(Q)$ is a locale.

Lemma

Let (Q, ς_Q) be an equivariantly supported A-A-quantale. Then, If ς_Q is equivariant then it is stable.

It is well-know that when Q is unital, the notion of equivariance and stability coincide when the base locale is $\downarrow(e)$. Nevertheless, in general this is not the case.

Lemma

Let (Q, ς) be an equivariantly supported A-A-quantale. Then, the sup-lattice maps ${}_{(.)}|1_Q : A \to Q$ and $\varsigma_Q : Q \to A$ are adjoints. Furthermore, the support ς_Q is unique and the map ${}_{(.)}|1_Q : A \to Q$ respects arbitrary meets.

Lemma

Let (Q, ς_Q) be an equivariantly supported A-A-quantale. Let $a, b \in Q$, and suppose that the following holds:

1. ${}_b|x = x$ *,*
2. ${}_b|1_Q \leq xx^*1_Q$ *.*

Then, $b = \varsigma_Q(x)$.

Lemma

Let (Q, ς_Q) be a supported Q_0-Q_0-quantale and (K, ς_K) be an equivariantly supported K_0-K_0-quantale. Then any homomorphism from Q to K preserves the support.

Remark

Having an equivariant support is a property of involutive quantales, rather than extra structure on it.

Quantal frames

Definition

By an A-A-quantal frame is meant an A-A-quantale Q such that for all $q, m_i \in Q$ and $a \in A$ the following properties hold:

$$q \wedge \bigvee_i m_i = \bigvee_i q \wedge m_i \tag{8}$$

$$({}_a|q) \wedge m =_a |(q \wedge m) \tag{9}$$

$$m \wedge (q|_a) = (q \wedge m)|_a \tag{10}$$

Theorem

The class of unital equivariantly supported quantales frames coincides with the class of stably supported quantal frames.

Principal quantales

Theorem (Definition of principal quantale)

Let (Q, ς) be an equivariantly supported A-A-quantal frame. The following conditions are equivalent:

1. $Q \cong R(Q) \otimes_{T(Q)} L(Q)$ *in the category of* ***Frm****.*
2. *For any frame S and any frame homomorphism $h : R(Q) \to S$ and $k : L(Q) \to S$ such that $h|_{T(Q)} = k|_{T(Q)}$, there is a unique frame homomorphism $t : Q \to S$ such that $t|_{R(Q)} = h$ and $t|_{L(Q)} = k$.*
3. *The triple $(Q, {}_{(.)}|1_Q, 1_Q|_{(.)})$ is the co-kernel pair of the frame inclusion $i : \{b \in A \mid {}_b|1_Q = 1_Q|_b\} \to A$.*

Let X be a set. The set of binary relations

$$Rel(X) = \wp(X \times X)$$

is a unital involutive quantale. The multiplication is the composition in the forward direction: $RS = S \circ R$ and the involution is reversal.
Then $Rel(X)$ is a principal quantale, because it is an equivariantly supported $\wp(X)$-$\wp(X)$-quantale frame with:

$${}_{U}|R = (U \times X) \cap R\,, \quad R|_{U} = (X \times U) \cap R\,, \quad \text{and}$$

$$\varsigma(R) = \{(x,x) | (x,y) \in R, \text{ for some } y \in X\}\,.$$

Here we have that $R(Rel(X)) = L(Rel(X)) \cong \wp(X)$, and $T(Rel(X)) = \{\emptyset, X\}$. Thus

$$Rel(X) \cong \wp(X) \otimes \wp(X)\,.$$

Reflexive quantal frames

Definition

By a reflexive quantal frame (Q, υ) will be meant an A-A-quantal frame equipped with a frame homomorphism $\upsilon : Q \to A$ satisfying, for all $a \in A$:

$$\upsilon({}_a|1_Q) = a = \upsilon(1_Q|_a)\,. \tag{11}$$

Lemma

Let (Q, ς, υ) be a stably (not necessarily equivariantly) supported A-A-quantale which is also a reflexive quantal frame. Then

$$\upsilon(a1_Q) = \varsigma(a)\,, \tag{12}$$

for all $a \in Q$.

Let (Q, ς, υ) be an equivariantly supported reflexive quantal frame over A. Let us define G_0 and G_1 as follows:

$$\mathcal{O}(G_0) = A \quad \text{and} \quad \mathcal{O}(G_1) = Q.$$

Consider the sup-lattice homomorphism $\delta : Q \to A$ defined by

$$\delta(a) = \varsigma(a)$$

Lemma

δ is the direct image $d_!$ of an open map $d : G_1 \to G_0$

We can now define a locale map $i : G_1 \to G_1$ by the condition $i^*(q) = q^*$ because the involution of Q is a frame isomorphism. Even more, we have $i \circ i = id_Q$ and $i_! = i^*$. Let us define an open map

$$r : G_1 \to G_0$$

by putting $r = d \circ i$.

Lemma

The tensor product $Q \otimes_A Q$ coincides with the pushout of the homomorphisms d^* and r^*.

Our candidate for the inclusion of units $u : G_0 \to G_1$ is $u^*(a) = \upsilon(a)$, for all $a \in Q$:

Lemma

Let (Q, ς, υ) be an equivariantly supported reflexive quantal frame, and G be its associated involutive localic graph:

$$G = \begin{tikzcd} G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "r"] \arrow[r, shift right=2, "d"'] & G_0 \arrow[l, "u" description] \end{tikzcd}.$$

Then G is an involutive reflexive open graph.

Multiplicative quantal frames

Lemma

The quantale multiplication has the following factorization in the category of sup-lattices.

$$\begin{tikzcd} Q \otimes Q \arrow[d, two heads, "\pi"'] \arrow[dr, "\mu"] & \\ Q \otimes_A Q \arrow[r, "\mu_A"'] & Q\,. \end{tikzcd}$$

By a *multiplicative quantal frame* is meant an A-A-quantale frame such that the right adjoint of μ_A preserves joins.

Remark

Any inverse quantal frame is multiplicative (Resende, 2007).

Theorem

Let (Q, ς) be a multiplicative equivariantly supported A-A-quantal frame. Then, the localic graph

$$\begin{tikzcd} G = G_2 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left, "r"] \arrow[r, shift right, "d"'] & G_0 \end{tikzcd} ,$$

where m is defined as follows:

$$m^*(a) = (\mu)_*(a) = \bigvee_{xy \leq a} x \otimes y \, .$$

is an involutive open semicategory.

Definition (Unit laws)

Let Q be a multiplicative equivariantly supported reflexive quantal frame (Q, ς, υ). We say that Q satisfies unit laws if moreover the following condition holds for all $a \in Q$:

$$\bigvee_{xy \leq a} ({}_{\upsilon(x)}|y) = a$$

Lemma

Let Q be a multiplicative equivariantly supported reflexive quantal frame that satisfies unit laws, and let G be its associated involutive localic graph:

$$G = G_2 \xrightarrow{m} G_1 \underset{d}{\overset{r}{\rightrightarrows}} G_0.$$

$$\begin{tikzcd} G = G_2 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "r"] \arrow[r, shift right=2, "d"'] & G_0 \arrow[l, "u" description] \end{tikzcd}$$

Then G is an open involutive category.

Definition (Groupoid quantales)

By a groupoid quantale Q will be meant a multiplicative equivariantly supported reflexive quantal frame that satisfies unit laws and moreover satisfies the following condition, which is referred to as the inverse law:

$$_{\upsilon(a)}|1_Q \leq \bigvee_{xx^* \leq a} x \,. \tag{13}$$

Theorem

Let Q be a groupoid quantale, and let G be its associated involutive localic graph:

$$\begin{tikzcd} G = G_2 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "r"] \arrow[r, shift right=2, "d"'] & G_0 \arrow[l, "u" description] \end{tikzcd}.$$

Then G is an open groupoid.

Remark

- *In the unital case the inclusion of units map* $u : G_0 \to G_1$ *is given by* $u^*(a) = a \wedge e$ *which is an open map of locales, because the frame homomorphism* $u^*(a) = a \wedge e$ *is the right adjoint of the sup-lattice inclusion* $\iota : \varsigma(Q) \to Q$*, whose direct image is* $u_! = \iota$*. Therefore it is possible to show the unit laws of an internal category in terms of direct images without appealing to the unit laws axiom.*
- *A unital involutive quantal frame* Q *satisfies inverse laws if and only if it is inverse, i.e.,* $\bigvee Q_{\mathcal{I}} = 1_Q$

Unital groupoid quantales

Theorem

The class of unital equivariantly supported reflexive quantal frames satisfying inverse laws corresponds with the class of inverse quantal frames

Theorem

Let $(Q, \varsigma, \upsilon, e)$ be an equivariantly supported reflexive quantal frame with a unit and satisfying inverse laws. Then, Q satisfies unit laws.

Remark

In general, the two axioms are independent!

Quantal groupoids

Recall that given a groupoid quantale Q, we denote its associated open groupoid by $\mathcal{G}(Q)$.

Theorem

Let G be an open groupoid. Then its associated quantale $\mathcal{O}(G)$ is a groupoid quantale.

Theorem

$\mathcal{G}(\mathcal{O}(G)) \cong G$ and $\mathcal{O}(\mathcal{G}(Q)) \cong Q$ for any open groupoid G and a groupoid quantale Q.

Effective equivalence relations (revisited)

Theorem (Q. and Resende, 2015)

Let Q be a groupoid quantale. Then Q is a principal quantale if and only if $\mathcal{G}(Q)$ is an open effective equivalence relation.

Étale-complete Groupoids

- The domain and range maps must be open surjections.
- There is a geometric morphism $\pi : Sh(G_0) \to \mathcal{B}(G)$ with $\pi^* : \mathcal{B}(G) \to Sh(G_0)$ the forgetful functor (that forgets the action).
- The groupoid G is étale-complete if the following square of toposes is a pullback:

$$\begin{tikzcd} Sh(G_1) \arrow[r, "r"] \arrow[d, "d"'] & Sh(G_0) \arrow[d, "\pi"] \\ Sh(G_0) \arrow[r, "\pi"'] & \mathcal{B}(G) \end{tikzcd}$$

Theorem (Q. and Resende, 2015)

Let Q be a principal groupoid quantale. Then the topos $B(\mathcal{G}(Q))$ is localic, and $\mathcal{G}(Q)$ is étale-complete.