Linear structures on locales

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I. Introduction

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6. Joint work with João Paulo Santos (IST):
   - Open quotients of trivial vector bundles; arXiv:1510.06329
   - Linear structures on locales, Theory Appl. Categ. 31 (2016) 502–541

II. Quotient vector bundles

Definition

By a **quotient vector bundle** is meant a triple (π, A, q) consisting of a "linear bundle" $\pi : E \to X$ on a topological space X, a topological vector space (TVS) A, and a continuous open surjection $q : A \times X \to E$ that makes the following commute:

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Any Fell bundle on an étale locally compact Hausdorff groupoid, with $A = C_r^*(E)$ and $q = \mathrm{eval}$.

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Notation: $E_x := \pi^{-1}(\{x\})$ is the **fiber** over x

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5. π "has enough sections": for all $e \in E_x$ there is $a \in A$ such that $\hat{a}(x) = e$.

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$\mathrm{Sub}\,A := \{\text{linear subspaces of } A\}$ $\quad$ $\mathrm{Max}\,A := \{\text{closed linear subspaces of } A\}$

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The **kernel map** $\kappa : X \to \mathrm{Sub}\,A$ is defined by

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1. $E_x \cong A/\kappa(x)$.
2. The fibers E_x are Hausdorff iff $\kappa : X \to \mathrm{Max}\,A$.
3. κ determines E and q uniquely up to isomorphisms.

II. Quotient vector bundles

Theorem

Let A be a TVS, X a topological space, and $\kappa : X \to \mathrm{Sub}\,A$ any map. Obtain a commutative diagram

$$\begin{tikzcd} A \times X \arrow[r, "q"] \arrow[dr, "\pi_2"'] & E \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

by constructing E as the quotient of $A \times X$ defined by

$$(a,x) \sim (b,y) \iff x = y \text{ and } a - b \in \kappa(x) \, .$$

The quotient map $q : A \times X \to E$ is open iff κ is continuous with respect the lower Vietoris topology of $\mathrm{Sub}\,A$.

DEFINITION

$\mathrm{Sub}A$ is the **classifying space** for QVBs with TVS A.
The QVB

$$UA := (\pi_A : UA \to \mathrm{Sub}A,\ A,\ q_A)$$

classified by the identity on $\mathrm{Sub}A$ is the **universal QVB for** A.

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Corollary

Any QVB (π, A, q) is the pullback of UA along the kernel map:

$$\begin{array}{ccc} E & \longrightarrow & UA \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_A} \\ X & \xrightarrow[\kappa]{} & \mathrm{Sub}A \end{array}$$

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2. Closed balls topology on $\mathrm{Max} A$ (A normed) classifies QVBs with continuous norm.
3. With A a Banach space and X Hausdorff, $\mathrm{Max} A$ classifies QVBs which are Banach bundles.

Let $\mathscr{A} = (\pi : E \to X,\ A,\ q)$ be a QVB:

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There is an adjunction between complete lattices:

$$\begin{tikzcd} \Omega(X) \arrow[r, bend right, "\gamma"'] \arrow[r, phantom, "\perp"] & \operatorname{Sub} A \arrow[l, bend right, "\sigma"'] \end{tikzcd}$$

$$\begin{aligned} \sigma(V) &= \bigcup_{a \in V} \operatorname{supp}^\circ \hat{a} \\ \gamma(U) &= \{a \in A \mid \operatorname{supp}^\circ \hat{a} \subset U\} \end{aligned}$$

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Example

If X is sober and the zero section is closed in E then the QVB is spectral.

III. Linearized locales

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E.g., Banach bundles and Fell bundles as before.

THEOREM

Let A be a locally convex space. Then $\mathrm{Max}A$ (with the lower Vietoris topology) is sober.

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Example

If A is locally convex the universal QVB (π, A, q) with Hausdorff fibers is not spectral, even though $\mathrm{Max}\,A$ is sober.

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Let $\triangle$ be a locale, and A a TVS, equipped with an adjunction

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The triple (A, σ, γ) is called a **linear structure** on $\triangle$, and $\mathfrak{A} = (\triangle, A, \sigma, \gamma)$ is a **linearized locale**.

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EXAMPLE

Every spectral vector bundle $\mathscr{A}$ yields a linearized locale $\Omega(\mathscr{A}) = (\triangle, A, \sigma, \gamma)$, with $\triangle = \Omega(X)$, as described earlier.

Conversely, every linearized locale $\mathfrak{A} = (\triangle, A, \sigma, \gamma)$ defines a QVB $\Sigma(\mathfrak{A}) = (\pi : E \to X, A, q)$ with $X = \Sigma(\triangle)$ and kernel map $\mathfrak{k} = \gamma|_X$.

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$\Sigma(\mathfrak{A})$ *is a spectral vector bundle.*

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Corollary

Correspondence

$$\begin{tikzcd} \textit{Spectral vector bundles} \arrow[r, shift left, "\Omega"] & \textit{Linearized locales} \arrow[l, shift left, "\Sigma"] \end{tikzcd}$$

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Composition is defined by $(f_\flat, f^*) \circ (g_\flat, g^*) = (f_\flat \circ g_\flat, g^* \circ f^*)$.

IV. Morphisms of QVBs

1. Let $\mathscr{A} = (\pi : E \to X,\ A,\ q)$ and $\mathscr{B} = (\rho : F \to Y,\ B,\ r)$ be QVBs.
2. A morphism $f : \mathscr{B} \to \mathscr{A}$ should "induce" a continuous linear map $f^* : A \to B$.

Definition

A **morphism** $f : \mathscr{B} \to \mathscr{A}$ is a pair $(f_\flat, f^*)$ such that

- $f_\flat : Y \to X$ is continuous;
- $f^* : A \to B$ is continuous and linear;
- $q(a,\ f_\flat(y)) = 0 \quad \Longrightarrow \quad r(f^*(a),\ y) = 0$ for all $a \in A$, $y \in Y$.

If the above is an equivalence we say that f is **strict**.

Composition is defined by $(f_\flat, f^*) \circ (g_\flat, g^*) = (f_\flat \circ g_\flat, g^* \circ f^*)$.

Obtain category $QVBun_\Sigma$ of spectral vector bundles.

- There is a uniquely defined continuous fiberwise linear map $f^\sharp$ that makes the following diagram commute:

$$\begin{tikzcd}
F & E \times_X Y \arrow[l, "f^\sharp"'] \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\mathrm{pb}"] & E \arrow[d, "\pi"] \\
& Y \arrow[r, "f_\flat"'] \arrow[from=1-1, "\rho"'] & X
\end{tikzcd}$$

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- If π and ρ are local homeomorphisms the pair $(f_\flat, f^\sharp)$ is equivalent to a cohomomorphism of sheaves.
- Different from a morphism of fiber bundles $\begin{array}{ccc} F & \xrightarrow{f_1} & E \\ \rho\downarrow & & \downarrow\pi \\ Y & \xrightarrow[f_0]{} & X \end{array}$

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satisfying, for all $V \in \mathrm{Sub}\, A$, the inclusion $\sigma_B(\bar{f}(V)) \subset \underline{f}^*(\sigma_A(V))$:

$$\begin{tikzcd}
\mathrm{Sub}\, A \arrow[r, "\mathrm{Sub}\bar{f}"] \arrow[d, "\sigma_A"'] \arrow[dr, phantom, "\geq"] & \mathrm{Sub}\, B \arrow[d, "\sigma_B"] \\
\triangle_A \arrow[r, "\underline{f}^*"'] & \triangle_B
\end{tikzcd}$$

Definition

(cont.)

Definition

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Equivalently (the right adjoint of $\mathrm{Sub}\,\bar{f}$ coincides with $\bar{f}^{-1}$), we have:

$$\begin{tikzcd}
\mathrm{Sub}\,A & & \mathrm{Sub}\,B \arrow[ll, "\bar{f}^{-1}"'] \\
\triangle_A \arrow[u, "\gamma_A"] & \leq & \triangle_B \arrow[ll, "f_*"] \arrow[u, "\gamma_B"']
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This defines the **category of linearized locales** *LinLoc*.

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\operatorname{Sub} A & & \operatorname{Sub} B \arrow[ll, "\bar{f}^{-1}"'] \\
\triangle_A \arrow[u, "\gamma_A"] \arrow[rr, phantom, "\leq" near start, shift left=8] & & \triangle_B \arrow[ll, "\underline{f}_*"] \arrow[u, "\gamma_B"']
\end{tikzcd}$$

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If the above commutation relations are strict the morphism $\mathfrak{f}$ is called **strict**.

V. Morphisms of linearized locales

Theorem

The correspondence between spectral vector bundles and linearized locales extends to an adjunction

$$\begin{tikzcd} \mathit{LinLoc} \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathit{QVBun}_{\Sigma} \arrow[l, bend left, "\Sigma"] \end{tikzcd}$$

(and to a restricted adjunction considering only strict morphisms).