Geometric realizations of tricategories

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The homotopy theory of higher categorical structures has become a relevant part of the machinery of algebraic topology and algebraic K-theory. In this talk we present some contributions to the study of classifying spaces for tricategories, with applications to the homotopy theory of categories, monoidal categories, bicategories, braided monoidal categories, and monoidal bicategories. Any tricategory characteristically has associated various simplicial or pseudo-simplicial objects. We explore the relationship amongst three of them: the pseudo-simplicial bicategory so-called *Grothendieck nerve of the tricategory*, the simplicial bicategory termed its *Segal nerve*, and the simplicial set called its *Street geometric nerve*, and it proves the fact that the geometric realizations of all of these possible candidate 'nerves of the tricategory' are homotopy equivalent. Any one of these realizations could therefore be taken as the classifying space of the tricategory. Our results provide coherence for all reasonable extensions to tricategories of Quillens definition of the classifying space of a category as the geometric realization of the categorys Grothendieck nerve. Many properties of the classifying space construction for tricategories may be easier to establish depending on the nerve used for realizations. For instance, by using Grothendieck nerves we state and prove the precise form in which the process of taking classifying spaces transports tricategorical coherence to homotopy coherence. Segal nerves allow us to obtain an extension to bicategories of the results by Mac Lane, Stasheff, and Fiedorowicz about the relation between loop spaces and monoidal or braided monoidal categories by showing that the group completion of the classifying space of a bicategory enriched with a monoidal structure is, in a precise way, a loop space. With the use of geometric nerves, we obtain genuine simplicial sets whose simplices have a pleasing geometrical description in terms of the cells of the tricategory and, furthermore, we obtain an extension of the results by Joyal, Street, and Tierney about the classification of braided categorical groups and their relationship with connected, simply connected homotopy 3-types, by showing that, via the classifying space construction, bicategorical groups are a convenient algebraic model for connected homotopy 3-types.

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