Asymmetric filter convergence and completeness

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Completeness for metric spaces is traditionally presented in terms of convergence of Cauchy sequences, and for uniform spaces in terms of Cauchy filters. Somewhat more abstractly, a uniform space is complete if and only if it is closed in every uniform space in which it is embedded, and so isomorphic to any space in which it is densely embedded. This is the approach to completeness used in the point-free setting, that is, for uniform and nearness frames: a nearness frame is said to be complete if every strict surjection onto it is an isomorphism.

An important aspect of our work is asymmetry. The fact that an asymmetric distance function has two natural underlying topologies leads to the study of bitopological spaces, and, in the point-free setting, to biframes. Quasi-uniformities and quasi-nearnesses on biframes provide appropriate structures with which to investigate uniform and nearness ideas in the asymmetric context.

In [1] a notion of completeness (called "quasi-completeness") was presented for quasi-nearness biframes in terms of suitable strict surjections being isomorphisms, and a quasi-completion was constructed for any quasi-nearness biframe.

A primary aim of this talk is to show that quasi-completeness can indeed be viewed in terms of the convergence of certain filters, namely, the regular Cauchy bifilters. Bifilters on biframes were first introduced in [2]; they were presented as filters on the total part, generated by their first and second parts. They can equally well be thought of as certain characteristic functions to the **2**-chain; replacing **2** by an arbitrary biframe T leads to the notion of a T-valued bifilter and it is this more general concept of a bifilter that is needed here. An important tool is an appropriate composition for bifilters, which in the symmetric case is, of course, a trivial matter.

In [1] we showed that the right adjoint of the join map from the downset biframe is the "universal" bifilter; loosely speaking, this means that any other bifilter on the same domain factors via the universal bifilter. Here we show that the right adjoint of the quasi-completion is the universal regular Cauchy bifilter and use this to prove the characterization of quasi-completeness mentioned above.

It is also of interest to drop the regularity condition and consider Cauchy bifilters in their own right. We construct the so-called Cauchy filter quotient for a biframe using a quotient of the downset biframe that involves only the Cauchy, and not the regularity, condition. Like the quasi-completion, this provides a universal Cauchy bifilter; unlike the quasi-completion, this construction is functorial. In order to obtain functoriality

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of the quasi-completion for certain biframes, we require a notion of having enough regular Cauchy bifilters; this is discussed in the paper to be presented by Dr A. Schauerte.

References

[1] J. Frith and A. Schauerte, *Quasi-nearnesses on biframes and their completions*, Quaestiones Math. 33(4) (2010) 507–530.

[2] J. Frith and A. Schauerte, *An asymmetric approach to filters in strict extensions and quotients*, Acta Math. Hungarica (2012) (doi:10.1007/s10474-012-0208-5).