

The relation of the so-called natural numbers to real mathematics

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Peano's 1890 space-filling curve was a more profound "crisis of foundations" than the famous crises of the 20th century, such as the alleged undecidability. However, Goedel's theorem has a hypothesis, namely: "If a system of natural numbers can be interpreted in a theory, then its set of true statements is not recursive." Noting that Peano's construction and later related ones seem to also depend on an abusive use of the natural numbers, I proposed in my Milan lectures around 1980 that powerful theories could be constructed which would avoid these impediments to mathematics and hence would employ a relationship, to the idealized totality of whole numbers, that inverts the relationship that (since Kronecker) has been complacently called "natural". Indeed, Dedekind's construction of an NNO by infinite intersection (verified by Freyd in any topos) clearly demonstrates a higher-order idealization than that of the continuum. Courage could be derived from Tarski's 1945 success; even though that was limited to real polynomials, it seemed to me to be extendable to stronger algebraic theories. Independently, around the same time Grothendieck in his 1984 "Esquisse d'un Programme" sought to rid cohomology theory of the pathologies deriving from the unwarranted intrusion into geometry of the many constructions similar to Peano's. Though a century of sermons had maintained that these monstrosities demonstrate the essential unreliability of intuition, Grothendieck boldly asked for a Tame Topology, whose features he elaborated.

The need is not for the banishment of N (nor for an incoherent philosophy such as finitism or predicativism), but the recognition that N constitutes a higher-order idealization. More precisely, two aspects need to be separated in order to lay bare their interaction: the real mathematics of objective space and quantity, vs. the metamathematics of our subjective struggles to master it; it is within this "metamathematics" that the idealization N occurs. Clearest will be a mathematical model of this interaction in a topos having a well-chosen site with a rich R such that the inclusion $N \rightarrow R$ exists but is not an equalizer of maps $R \rightarrow R^n$. To that end I suggested in the Proceedings of Como 90 that a categorical clarification be developed of the work of those logicians who, starting independently in the early 80s, have succeeded to expand Tarski's 1945 result, thus providing a rigorous partial solution that suggests what sort of richness can be achieved in such a model.