Monadic actions

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A monoid M in a monoidal category $(\mathsf{C}, \otimes, E)$ gives rise to a monad whose Eilenberg-Moore category C^M is the category of M-actions and equivariant maps—a fact that is well-known in the case of M-actions in Set. Further examples, provided by the category of R-modules or the category of a quantale action on sup-semilattices, suggest the following general principle:

> The category $(\mathsf{C}^{\mathbb{T}})^M$ of actions of a monoid M in a monoidal Eilenberg–Moore category $\mathsf{C}^{\mathbb{T}}$ is monadic over C.

However, before being able to define actions in $\mathsf{C}^{\mathbb{T}}$, we need a tensor $\circledast$ on $\mathsf{C}^{\mathbb{T}}$ that encodes the "bilinear" nature of the action morphism $M \circledast X \to X$: this structure emerges in the form of a monoidal monad on $(\mathsf{C}, \otimes, E)$ that facilitates the definition of C-morphisms that are "$\mathbb{T}$-algebra homomorphisms in each variable", as originally suggested in [1] and studied in [2].

References

[1] F.E.J. Linton. *Coequalizers in categories of algebras.* In Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), pages 75–90. Springer, Berlin, 1969.

[2] R. Guitart. *Tenseurs et machines.* Cahiers Topologie Géom. Différentielle, 21(1):5–62, 1980.