Many for the price of one duality principle
for variety-based topological spaces

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In [2], D. Hofmann considered topological spaces as generalized orders, characterizing those ones, which satisfy a suitably defined topological analogue of the complete distributivity law. He showed that the category of distributive spaces is dually equivalent to a certain category of frames, observing that their both represent the idempotent split completion of the same category. The results are based in four submonads of the filter monad $\mathbb{F}$ on the category Top of topological spaces [1]. In the talk, we lift the duality of [2] to the setting of lattice-valued topological spaces [3].

Given a variety of algebras $\mathbf{A}$, its reduct $(\mathbf{B}, \| - \|)$, and an $\mathbf{A}$-algebra $A$, consider the category $\mathbf{A} \text{-Top}$ of $\mathbf{A}$-topological spaces ($\mathbf{A}$-spaces), whose objects are pairs $(X, \tau)$, with $X$ a set and $\tau$ an $\mathbf{A}$-subalgebra of the powerset algebra $A^X$, and whose morphisms $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are maps $X_1 \xrightarrow{f} X_2$ with $f^*_\tau (\alpha) = \alpha \circ f \in \tau_1$ for every $\alpha \in \tau_2$ [6].

There exists a functor $\mathbf{A} \text{-Top} \xrightarrow{\mathcal{O}_A} \mathbf{B}^{op}$, $\mathcal{O}_A((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = \| \tau_1 \| \xrightarrow{(f^*)^{op}} \| \tau_2 \|$, which has a right adjoint [5], thereby providing a monad $\mathbb{T}_A$ on $\mathbf{A} \text{-Top}$.

Let $\mathbf{A} \text{-Top}_0$ be the full subcategory of $\mathbf{A} \text{-Top}$ of $T_0$ $\mathbf{A}$-spaces, i.e., $\mathbf{A}$-spaces $(X, \tau)$, where every distinct $x_1, x_2 \in X$ have $\alpha \in \tau$ with $\alpha(x_1) \neq \alpha(x_2)$. There exists the restriction $\mathbb{T}_A^0$ of the monad $\mathbb{T}_A$ to $\mathbf{A} \text{-Top}_0$. If $\mathbf{B}$ is enriched in the category $\mathbf{Pos}$ of posets, one defines a preorder on an $\mathbf{A}$-space $(X, \tau)$ by $x_1 \sqsubseteq x_2$ iff $\alpha(x_1) \leqslant \alpha(x_2)$ for every $\alpha \in \tau$, which is an order on $T_0$ $\mathbf{A}$-spaces (thereby providing a functor $\mathbf{A} \text{-Top}_0 \xrightarrow{\mathcal{S}pec} \mathbf{Pos}$). For some $\mathbf{A}$ and $\mathbf{B}$, one gets that $\mathbb{T}_A^0$ is of Kock-Zöberlein type [1].

Let $((X, \tau), h)$ be the Eilenberg-Moore category of $\mathbb{T}_A^0$. By [2], a $\mathbb{T}_A^0$-algebra $((X, \tau), h)$ is called $\mathbb{T}_A^0$-distributive provided that $h$ has a left adjoint (in the sense of posets) $(X, \tau) \xrightarrow{\bot} \mathbb{T}_A^0(X, \tau)$ in $\mathbf{A} \text{-Top}_0$ (which is then a $\mathbb{T}_A^0$-homomorphism with $h \circ t = 1_{(X, \tau)}$). $\mathbf{Spl}(\mathbf{A} \text{-Top}_0)^{\mathbb{T}_A^0}$ is the full subcategory of $\mathbf{A} \text{-Top}_0^{\mathbb{T}_A^0}$ of $\mathbb{T}_A^0$-distributive $\mathbb{T}_A^0$-algebras. Moreover, a $\mathbf{B}$-algebra $B$ is called $\mathbf{A}$-spatial provided that every $b_1, b_2 \in B$ with $b_1 \not\leq b_2$ have $p \in \mathbf{B}(B, \| A \|)$ with $p(b_1) \not\leq p(b_2)$. $B$ is called a $\mathbf{B}$-frame provided that it has a $\lor$-semilattice reduct, and its primitive operations with non-zero arities distribute over $\lor$. $\mathbf{B} \text{- Frm}$ is the full subcategory of $\mathbf{B}$ of $\mathbf{A}$-spatial $\mathbf{B}$-frames.

Following [2], we describe the objects of $\mathbf{Spl}(\mathbf{A} \text{-Top}_0^{\mathbb{T}_A^0})$ and $\mathbf{B} \text{- Frm}$, and show that the categories are dually equivalent. In particular, one gets the dualities of [2].

* This research was supported by ESF Project 2009/0223/1DP/1.1.1.2.0/09/APIA/VIAA/008 of the University of Latvia.
References


