Closed morphisms via neighbourhood operators

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- A general theory of neighbourhood operators was introduced by D. Holgate and J. Šlapal (2011) on a category **C** equipped with a $(\mathcal{E}, \mathcal{M})$-factorisation system:

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(N1) If $p \in \nu_X(m)$ and $p \leq q$, then $q \in \nu_X(m)$;

(N2) If $n \in \nu_X(m)$, then $m \leq n$;

(N3) If $m \leq n$ then $\nu_X(n) \subseteq \nu_X(m)$;

(N4) If $f : X \to Y$ is in $\mathbf{C}$ and $k \in \nu_Y(n)$, then $f^{-1}[k] \in \nu_X(f^{-1}[n])$.

(N5) If $\mathcal{G} \subseteq \mathcal{M}/X$ and for any $g \in \mathcal{G}$, $m \in \nu_X(g)$, then $m \in \nu_X(\bigvee \mathcal{G})$.

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Notation

For any $f : X \to Y$ in $\mathbf{C}$, $\mathcal{G} \subseteq \mathcal{M}/X$ and $\mathcal{H} \subseteq \mathcal{M}/Y$, denote:

$$f[\mathcal{G}] :=\uparrow \{f[g] \mid g \in \mathcal{G}\}$$

and

$$f^{-1}[\mathcal{H}] :=\uparrow \{f^{-1}[h] \mid h \in \mathcal{H}\}$$

Thus

$$\text{(N4)} \ \simeq \ \ f^{-1}[\nu_Y(n)] \subseteq \nu_X(f^{-1}[n]).$$

Remark

- (N5) holds in $\mathbf{C} = \mathbf{Top}$ with the usual neighbourhood system since a union of open sets is open.
  Is it essential?
- Rather consider very basic properties of neighbourhoods and eliminate (N5).

$\Rightarrow$ ν is a neighbourhood operator if it satisfies (N1), (N2), (N3) and (N4). We call ν a *regular neighbourhood operator* if it satisfies (N5) in addition.

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Facts

$NBH(\mathbf{C}, \mathcal{M})$ denotes the category of neighbourhood operators with natural inclusion and $RNBH(\mathbf{C}, \mathcal{M})$ denotes that of the regular neighbourhood operators.

Proposition

1. *$RNBH(\mathbf{C}, \mathcal{M})$ is reflective subcategory of $NBH(\mathbf{C}, \mathcal{M})$;*
2. *$RNBH(\mathbf{C}, \mathcal{M})$ is equivalent to the category of the so-called interior operators.*

Interior operators were first introduced by S. R. J Vorster (2000).

Definition

(*Castellini, 2011; Ochoa and Luna-Torres, 2010*) An interior operator i on $\mathbf{C}$ is a family $(i_X)_{X\in\mathbf{C}}$ with $i_X : \mathcal{M}/X \to \mathcal{M}/X$ and such that:

(I1) For any $m \in \mathcal{M}/X$, $i_X(m) \leq m$;

(I2) If $m \leq n$, then $i_X(m) \leq i_X(n)$;

(I3) If $f : X \to Y$ is in $\mathbf{C}$ and $n \in \mathcal{M}/Y$, then $f^{-1}[i_Y(n)] \leq i_X(f^{-1}[n])$.

- How do we treat the notion of closedness with respect to a concept which is thought to be natural for openness?
- Compactness and separation have been extensively studied in categories on which were given some notion of closure (cf. M.M. Clementino, E. Giuli and W. Tholen, *A Functional Approach to General Topology*, 2004). The idea of defining these notions by requiring the diagonal map $X \to X \times X$ and the terminal map $X \to \mathbf{1}$ to be closed could be traced back to Penon (1972) and Manes (1974).

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Definition

A map $f : X \to Y$ is said to be ν-closed, where $\nu \in NBH(\mathbf{C}, \mathcal{M})$, if for any $n \in \mathcal{M}/Y$

$$f^{-1}[\nu_Y(n)] = \nu_X(f^{-1}[n]).$$

f is closed if for any $n \in \mathcal{M}/Y$ and $k \in \nu_X(f^{-1}[n])$, there is $p \in \nu_Y(n)$ such that $f^{-1}[p] \leq k$.

Remarks

- Recall that a map $f : X \to Y$ is closed with respect to a closure operator c if for any $m \in \mathcal{M}/X$, $f[c_X(m)] \cong c_Y(f[m])$;
  This "symmetry" is present in other notions:
- We say that f is ν-open if $f[\nu_X(m)] = \nu_Y(f[m])$;
- We say that f is ν-initial if $\nu_X(m) = f^{-1}[\nu_Y(f[m])]$;
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Given $\nu \in NBH(\mathbf{C}, \mathcal{M})$, let $\mathcal{F}(\nu) := \{f \mid f\ \nu - \text{closed}\ \}$

Proposition

1. *$\mathcal{F}(\nu)$ contains isomorphisms and is closed under composition;*
2. *If $gf \in \mathcal{F}(\nu)$ and $f \in \mathcal{E}'$, where $\mathcal{E}'$ is the class in $\mathcal{E}$ stable under pullback along morphisms in $\mathcal{M}$, then $g \in \mathcal{F}(\nu)$.*

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- *If $m : M \to X \in \mathcal{F}(\nu) \cap \mathcal{M}$, then m is ν-initial and satisfies the property that if $m \wedge k = 0_X$ for any $k \in \mathcal{M}/X$, then there is $n \in \nu_X(k)$ such that $m \wedge n = 0_X$;*
- *In the following pullback diagram:*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

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What makes $\mathcal{F}(\nu)$ special?

Definition

An object $X \in \mathbf{C}$ is ν-compact if for any $Y \in \mathbf{C}$, the projection $p_Y : X \times Y \to Y$ belongs to $\mathcal{F}(\nu)$.

In **Top**, this is sometimes called the Tube's Lemma: " *A space X is compact iff for any space Y and $y \in Y$, if O an open set containing $X \times \{y\}$, then there is a neighbourhood N of y, such that $X \times \{y\} \subseteq X \times N \subseteq O$.*"

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- A family $\mathcal{A} = \{f_i : X \to X_i \mid i \in I\}$ is said to be ν-initial if for any $m \in \mathcal{M}/X$,
$$\nu_X(m) = \cup\{f_i^{-1}[\nu_i(f_i[m])] \mid i \in I\}.$$

(If $\mathcal{A}$ is c-initial for a closure operator c, what would that mean?)

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Theorem

Let $X = \prod_I X_i$ *and* $\nu \in NBH(\mathbf{C}, \mathcal{M})$. *Assume that:*

- $\{p_{X_i} : X \to X_i \mid i \in I\} \subseteq \mathcal{E}^*$ *and is* ν*-initial;*
- *Every natural projection* $\{\pi_j : P \to P_j \mid j \in J\}$ *is* ν*-initial for any finite* J*;*

Then if every X_i *is* ν*-compact, then so is* X.

Definition

(Day-Kelly,1970) A topological space X is exponentiable if and only if for every neighbourhood U of a point $x \in X$ there is a smaller neighbourhood U of x such that every open cover of U admits a finite subcover of V.

Theorem

(E. Colebunders and G. Richter, 2001) The product functor $X \times - : \mathbf{Top} \to \mathbf{Top}$ preserves quotients, if X is quasi-locally compact.

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If $V << U$ in X and $y \in Y$, then for any open set W containing $U \times \{y\}$, there is an open rectangle $S \times T \subseteq W$ such that $V \times \{y\} \subseteq S \times T$.

Kuratowski-Mrówka type theorem?

Definition

Let $m : V \to U$ be an embedding in **Top**. We say that V is *relatively compact* with respect to U if for any space X, in the following diagram:

$$
\begin{tikzcd}
U \times X \arrow[dr, "\pi"] & \\
 & X \\
V \times X \arrow[uu, "m \times 1_Y"] \arrow[ur, "p"'] &
\end{tikzcd}
$$

we have $(m \times 1_Y)[p^{-1}[\nu_Y(x)]] = \nu_{U\times Y}(\pi^{-1}[x])$.

Happy Birthday!