A push forward construction and the comprehensive factorization for internal crossed modules

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(joint work with S. Mantovani and G. Metere)

Workshop on Category Theory
in honour of George Janelidze, on the occasion of his 60th birthday
Coimbra, July 13, 2012
Introduction
Let $\mathbb{A}$ be an abelian category, $A$ and $C$ objects of $\mathbb{A}$. 
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$$c^* : \text{EXT}(C, A) \to \text{EXT}(C', A)$$
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Any morphism $c: C' \to C$ determines a functor

$$c^*: \text{EXT}(C, A) \to \text{EXT}(C', A)$$

by means of the pullback along $c$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & \downarrow & & \downarrow \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$
Let $\mathcal{A}$ be an abelian category, $A$ and $C$ objects of $\mathcal{A}$. Short exact sequences with kernel $A$ and cokernel $C$ form a groupoid $\text{EXT}(C, A)$. Equivalence classes form an abelian group:

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Any morphism $c : C' \to C$ determines a functor

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& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

And this gives a group homomorphism

$$\text{Ext}(C, A) \to \text{Ext}(C', A).$$
Dually, any morphism $a: A \to A'$ determines a functor:

$$a_*: \text{EXT}(C, A) \to \text{EXT}(C, A')$$
Dually, any morphism \( a: A \rightarrow A' \) determines a functor:

\[
a_* : \text{EXT}(C, A) \rightarrow \text{EXT}(C, A')
\]

by means of the pushout along \( a \):

\[
\begin{array}{cccccc}
0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \mid & & \mid \\
0 & \rightarrow & A' & \rightarrow & B' & \rightarrow & C & \rightarrow & 0
\end{array}
\]
Dually, any morphism $a: A \to A'$ determines a functor:

$$a_*: \text{EXT}(C, A) \to \text{EXT}(C, A')$$

by means of the pushout along $a$:

\[
\begin{array}{c}
0 \to A \to B \to C \to 0 \\
\downarrow^a \quad \downarrow \quad \downarrow \\
0 \to A' \to B' \to C \to 0
\end{array}
\]

And this gives a group homomorphism

$$\text{Ext}(C, A) \to \text{Ext}(C, A').$$
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$$\phi: G \times A \rightarrow A$$
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Short exact sequences inducing the same action of $G$ on $A$ form a groupoid $OPEXT(G, A, \phi)$. 
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$$
Opext(G, A, \phi) \cong H^2_\phi(G, A).
$$
The non-abelian setting is more complicated. Example: groups. Any short exact sequence of abelian kernel $A$ and cokernel $G$ determines an action of $G$ on $A$:

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Short exact sequences inducing the same action of $G$ on $A$ form a groupoid $\text{OPEXT}(G, A, \phi)$. Equivalence classes form an abelian group:

$$\text{Opext}(G, A, \phi) \cong H^2_{\phi}(G, A).$$

Again, for any group homomorphism $g: G' \rightarrow G$, the pullback construction determines a functor:

$$g^*: \text{OPEXT}(G, A, \phi) \rightarrow \text{OPEXT}(G', A, g^*(\phi)),$$
where $g^*(\phi)$ is given by the composite:

\[ G' \times A \longrightarrow \longrightarrow \longrightarrow A \]

\[ \downarrow g \times 1 \quad \downarrow \phi \]

\[ G \times A \]
where $g^*(\phi)$ is given by the composite:

\[
\begin{array}{cccc}
G' \times A & \xrightarrow{g^*(\phi)} & A \\
g \times 1 & \downarrow & \\
G \times A & \xleftarrow{\phi} & \\
\end{array}
\]

And again a group homomorphism:

\[
H^2_\phi(G, A) \to H^2_{g^*(\phi)}(G', A).
\]
The pushout construction no longer works.
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Problems:
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  - the pushout of a normal mono is not a normal mono in general;
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- the pushout of a normal mono is not a normal mono in general;
- a morphism $a: A \rightarrow A'$ does not determine an action of $G$ on $A'$ in a canonical way.
The pushout construction no longer works. Problems:

- the pushout of a normal mono is not a normal mono in general;
- a morphism \( a: A \to A' \) does not determine an action of \( G \) on \( A' \) in a canonical way.

So we need an action of \( G \) on \( A' \):

\[
\phi': G \times A' \to A'
\]
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Problems:

- the pushout of a normal mono is not a normal mono in general;
- a morphism $a: A \to A'$ does not determine an action of $G$ on $A'$ in a canonical way.

So we need an action of $G$ on $A'$:

$$\phi': G \times A' \to A'$$

and we require that $a$ is equivariant, i.e.:

$$
\begin{array}{ccc}
G \times A & \xrightarrow{\phi} & A \\
1 \times a & \downarrow & a \\
G \times A' & \xrightarrow{\phi'} & A'
\end{array}
$$
These data allow to construct the so called *push forward* along $a$:

\[
\begin{array}{c}
0 \longrightarrow A \\ a \downarrow \\
0 \longrightarrow A'
\end{array}
\quad
\begin{array}{c}
\quad k \downarrow p.f. \quad f \downarrow e \\
E \longrightarrow G \longrightarrow 0
\end{array}
\quad
\begin{array}{c}
0 \longrightarrow E' \\
0 \longrightarrow E'
\end{array}
\quad
\begin{array}{c}
\quad k' \downarrow f' \\
G \longrightarrow 0
\end{array}
\]
These data allow to construct the so called *push forward* along $a$:

$$
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow & & \downarrow \text{p.f.}
\end{array}
\begin{array}{ccc}
E & \rightarrow & G \\
\downarrow & & \downarrow e
\end{array}
\begin{array}{ccc}
0 & \rightarrow & 0
\end{array}
\begin{array}{ccc}
0 & \rightarrow & A' \\
\downarrow & & \downarrow \text{p.f.}
\end{array}
\begin{array}{ccc}
E' & \rightarrow & G \\
\downarrow & & \downarrow f'
\end{array}
\begin{array}{ccc}
0 & \rightarrow & 0
\end{array}

which determines a functor:

$$
a_* : \text{OPEXT}(G, A, \phi) \rightarrow \text{OPEXT}(G, A', \phi')
$$
These data allow to construct the so called *push forward* along $a$:

$$
\begin{array}{c}
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\
& & \downarrow a & & \downarrow p.f. & & \downarrow e & & \\
0 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 0
\end{array}
$$

which determines a functor:

$$
a_* : \text{OPEXT}(G, A, \phi) \to \text{OPEXT}(G, A', \phi')
$$

and a group homomorphism:

$$
H_\phi^2(G, A) \to H_{\phi'}^2(G, A').
$$
Construction of the push forward (for groups):

\[
\begin{array}{c}
\text{A} \\
\text{a}
\end{array} \quad \xrightarrow{k} \quad \begin{array}{c}
E \\
i_E
\end{array} \quad \xrightarrow{i_E} \quad \begin{array}{c}
E \rtimes_{f^* (\phi')} A' \\
q
\end{array} \quad E'
\]

where \( q = \text{coeq}(i_E k, i_{A'} a) \).
Introduction
A push forward construction
A particular case

Construction of the push forward (for groups):

\[
\begin{array}{ccc}
A & \xrightarrow{k} & E \\
\downarrow{a} & & \downarrow{i_E} \\
A' & \xrightarrow{i_{A'}} & E \rtimes_{f^* (\phi')} A' \\
\end{array}
\xrightarrow{q} E'
\]

where \( q = \text{coeq}(i_E k, i_{A'} a) \).

Universal property:

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow{a} & & \downarrow{p.f.} \\
0 & \rightarrow & A' \\
\end{array}
\xrightarrow{k'} \begin{array}{ccc}
E & \rightarrow & G & \rightarrow & 0 \\
\downarrow{f} & & \downarrow{f'} & & \\
E' & \rightarrow & G & \rightarrow & 0 \\
\end{array}
\]
Construction of the push forward (for groups):

\[
\begin{array}{ccc}
A & \xrightarrow{k} & E \\
\downarrow a & \quad & \downarrow i_E \\
A' & \quad & E \rtimes_{f^*(\phi')} A' \\
\end{array}
\xrightarrow{q} E'
\]

where \( q = \text{coeq}(i_E k, i_{A'} a) \).

Universal property:

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow a & \quad & \downarrow p.f. \\
0 & \rightarrow & A' \\
\end{array}
\xrightarrow{0} \quad
\begin{array}{ccc}
0 & \rightarrow & A' \\
\downarrow k' & \quad & \downarrow k'' \\
0 & \rightarrow & A' \\
\end{array}
\xrightarrow{0} \quad
\begin{array}{ccc}
0 & \rightarrow & E \\
\downarrow f & \quad & \downarrow f' \\
0 & \rightarrow & G \\
\end{array}
\xrightarrow{0} \quad
\begin{array}{ccc}
0 & \rightarrow & E' \\
\downarrow f'' & \quad & \downarrow g \\
0 & \rightarrow & G \\
\end{array}
\xrightarrow{0} \quad
\begin{array}{ccc}
0 & \rightarrow & E'' \\
\downarrow f' & \quad & \downarrow f'' \\
0 & \rightarrow & G \\
\end{array}
\xrightarrow{0}
\]
Construction of the push forward (for groups):

$\xymatrix{ & E \ar[rr]^{i_E} \ar[dl]_k \ar[dr]^{i_{A'}} & & A' \ar[dl]_a \ar[dr]^{i_{A'}} \ar[dd]_{q = \text{coeq}(i_E k, i_{A'} a)} & \ar[r]^q & E' \ar[dl]_i \ar[dr]_{E \rtimes f^*(\phi')} & & }$

Universal property:
Questions:
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- Is there a push forward construction for extensions with non-abelian kernel?
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- Is there an internal version of this construction?
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- Is there a push forward construction for extensions with non-abelian kernel?
- Is there an internal version of this construction?
- Can it be extended to crossed modules?
A push forward construction in semi-abelian categories
Let $\mathcal{C}$ be a semi-abelian category.
Let $\mathcal{C}$ be a semi-abelian category. A precrossed module in $\mathcal{C}$ is a morphism

$$\partial: H \to H_0,$$

together with an internal action

$$\xi: H_0 \lhd H \to H,$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
H_0 \lhd H & \xrightarrow{\xi} & H \\
\downarrow 1 \lhd \partial & & \downarrow \partial \\
H_0 \lhd H_0 & \xrightarrow{\chi} & H_0
\end{array}
$$
Let $\mathcal{C}$ be a semi-abelian category.

A precrossed module in $\mathcal{C}$ is a morphism

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$$\xi : H_0 \lrcorner H \to H,$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
H_0 \lrcorner H & \xrightarrow{\xi} & H \\
1 \lrcorner \partial \downarrow & & \downarrow \partial \\
H_0 \lrcorner H_0 & \xrightarrow{\chi} & H_0
\end{array}
$$

If we want $\partial$ to be a crossed module, we need a further condition, which is not in general the straightforward generalization of the Peiffer condition for crossed modules of groups.
G. Janelidze in '03 gave a definition of internal crossed module, showing the equivalence with internal groupoids.
G. Janelidze in '03 gave a definition of internal crossed module, showing the equivalence with internal groupoids. However, if $\mathcal{C}$ satisfies the “Smith is Huq” property, the Peiffer condition:

\[
\begin{array}{c}
H \downarrow \downarrow \\
\partial \downarrow \\
H_0 \downarrow \\
\end{array}
\]

\[
\begin{array}{c}
\Rightarrow \\
\chi \\
\xi \\
\end{array}
\]

\[
\begin{array}{c}
H \\
H_0 \\
\end{array}
\]

turns out to be sufficient to characterize internal crossed modules among precrossed modules (Martins-Ferreira and Van der Linden, '10).
Hartl (unpublished preprint ’10): push forward of a normal monomorphism in semi-abelian setting with conditions expressed in terms of cross effects;
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He should have presented here a generalization of his result to crossed modules;
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- He should have presented here a generalization of his result to crossed modules;

- Meanwhile we reinterpreted conditions in terms of internal actions and semi-direct products, obtaining, for push forward of (pre)crossed modules, an equivalent result.
Let $\mathcal{C}$ be a semi-abelian category, $\partial$ and $p$ two morphisms in $\mathcal{C}$:

$$
\begin{array}{c}
H \\ p
\end{array}
\begin{array}{c}
\rightarrow \\
p
\end{array}
\begin{array}{c}
H_0 \\ G
\end{array}
$$

satisfying the following conditions:
Let $\mathcal{C}$ be a semi-abelian category, $\partial$ and $p$ two morphisms in $\mathcal{C}$:

$$
\begin{align*}
H & \xrightarrow{\partial} H_0 \\
p & \downarrow \\
G & 
\end{align*}
$$

satisfying the following conditions:

1) there is an action $\xi : H_0 \triangleright H \to H$ such that $(\partial, \xi)$ is a precrossed module;
Let $\mathcal{C}$ be a semi-abelian category, $\partial$ and $p$ two morphisms in $\mathcal{C}$:

\[
\begin{array}{c}
H \xrightarrow{\partial} H_0 \\
p \\
\downarrow \\
G
\end{array}
\]

satisfying the following conditions:

1) there is an action $\xi : H_0 \ltimes H \rightarrow H$ such that $(\partial, \xi)$ is a precrossed module;

2) there is an action $\alpha : H_0 \ltimes G \rightarrow G$, and $p$ is equivariant:

\[
\begin{array}{c}
H_0 \ltimes H \xrightarrow{\xi} H \\
1 \ltimes p \\
\downarrow \\
H_0 \ltimes G \xrightarrow{\alpha} G \\
p \\
\downarrow
\end{array}
\]
3) the following diagram commutes:

\[
\begin{array}{ccc}
(H \rtimes_{\xi} H_0)bG & \xrightarrow{\varphi^bG^1} & H_0bG \\
(p \rtimes 1)b1 & \downarrow & \downarrow \alpha \\
(G \rtimes_{\alpha} H_0)bG & \xrightarrow{\chi} & G
\end{array}
\]
3) the following diagram commutes:

\[
\begin{array}{c}
(H \rtimes_\xi H_0) \rtimes G & \xrightarrow{\varphi^{b1}} & H_0 \rtimes G \\
(p \times 1) \rtimes 1 & \downarrow & \downarrow \\
(G \rtimes_\alpha H_0) \rtimes G & \xrightarrow{\chi} & G
\end{array}
\]

where \( \varphi \) is defined by the universal property of semi-direct product:

\[
\begin{array}{c}
H \xrightarrow{i_H} H \rtimes_\xi H_0 \xleftarrow{i_{H_0}} H_0 \\
\partial & \downarrow \varphi & \downarrow 1 \\
H_0 & \downarrow & H_0
\end{array}
\]
3) the following diagram commutes:

\[
\begin{array}{ccc}
(H \rtimes_{\xi} H_0) \rtimes G & \xrightarrow{\varphi^b} & H_0 \rtimes G \\
(p \rtimes 1) \rtimes 1 & \downarrow & \downarrow {\alpha} \\
(G \rtimes_{\alpha} H_0) \rtimes G & \xrightarrow{\chi} & G
\end{array}
\]

where \(\varphi\) is defined by the universal property of semi-direct product:

\[
\begin{array}{ccc}
H & \xrightarrow{i_H} & H \rtimes_{\xi} H_0 \\
\downarrow {\partial} & & \downarrow \varphi \\
H_0 & & \downarrow 1 \\
\end{array}
\]

These conditions are sufficient to obtain a push forward construction.
Theorem

There exists an object $G \rtimes^H H_0$, together with a crossed module $	ilde{\partial}: G \to G \rtimes^H H_0$, with $\text{coker}(\tilde{\partial}) \cong \text{coker}(\partial)$, and a morphism $\tilde{\rho}_0: H_0 \to G \rtimes^H H_0$, such that the following diagram is a morphism of precrossed modules:

\[
\begin{array}{ccc}
H & \xrightarrow{\partial} & H_0 \\
\downarrow p & & \downarrow \tilde{\rho}_0 \\
G & \xrightarrow{\tilde{\partial}} & G \rtimes^H H_0
\end{array}
\]
Theorem

There exists an object $G \rtimes^H H_0$, together with a crossed module
$\tilde{\partial}: G \to G \rtimes^H H_0$, with $\text{coker}(\tilde{\partial}) \cong \text{coker}(\partial)$, and a morphism
$\tilde{p}_0: H_0 \to G \rtimes^H H_0$, such that the following diagram is a morphism of
precrossed modules:

![Diagram]

which is universal in the following sense: for any other morphism $(p, p_0)$
of precrossed modules, where $(\tilde{\partial}', \xi')$ is a crossed module and $p_0^*(\xi') = \alpha,$
There exists an object $G \times^H H_0$, together with a crossed module $\tilde{\partial} : G \to G \times^H H_0$, with $\text{coker}(\tilde{\partial}) \cong \text{coker}(\partial)$, and a morphism $\tilde{p}_0 : H_0 \to G \times^H H_0$, such that the following diagram is a morphism of precrossed modules:

![Diagram](image)

which is universal in the following sense: for any other morphism $(p, p_0)$ of precrossed modules, where $(\partial', \xi')$ is a crossed module and $p_0^*(\xi') = \alpha$, there exists a unique factorization $t$, with $t \tilde{p}_0 = p_0$ and $(1_G, t)$ a morphism of crossed modules.
A remark about the notation.
A remark about the notation. If $H = 0$, conditions 1)–3) reduce to the request of existence of the action $\alpha$, and the above construction is nothing but semi-direct product:

\[
\begin{array}{ccc}
0 & \rightarrow & H_0 \\
\downarrow & & \downarrow i_{H_0} \\
G & \xrightarrow{i_G} & G \rtimes_\alpha H_0
\end{array}
\]
A remark about the notation. If $H = 0$, conditions 1)–3) reduce to the request of existence of the action $\alpha$, and the above construction is nothing but semi-direct product:

$$
\begin{array}{c}
0 \\ G \\
\downarrow \quad i_G \\
G \\
\downarrow \quad i_{H_0} \\
G \times_{\alpha} H_0 \\
\downarrow \partial' \quad t \\
G_0
\end{array}
$$

The universal property reduces to the universal property of semi-direct product.
If the category $\mathcal{C}$ is moreover action accessible (e.g. groups, Lie algebras, rings, any category of interest), we can replace condition 3) with the following condition:
If the category $\mathcal{C}$ is moreover action accessible (e.g. groups, Lie algebras, rings, any category of interest), we can replace condition 3) with the following condition:

$$3') \quad H_\bullet G \xrightarrow{\partial_1} H_0 \bullet G$$

$$\xymatrix{ H_\bullet G \ar[d]^{p \circ 1} \ar[r]^{\partial_1} & H_0 \bullet G \ar[d]^\alpha \\
G \bullet G \ar[r]_-\chi & G}$$

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A push forward construction
If the category $\mathcal{C}$ is moreover action accessible (e.g. groups, Lie algebras, rings, any category of interest), we can replace condition 3) with the following condition:

$$3') \quad H_\flat G \xrightarrow{\partial_\flat} H_0\flat G$$

$$p_\flat 1 \downarrow \quad \alpha \downarrow$$

$$G_\flat G \xrightarrow{\chi} G$$

which is in fact weaker:

$$H_\flat G \xrightarrow{i_{H_\flat 1}} (H \rtimes_\xi H_0)\flat G \xrightarrow{\varphi_\flat 1} H_0\flat G$$

$$p_\flat 1 \downarrow \quad (p \times 1)_\flat 1 \downarrow \quad \alpha \downarrow$$

$$G_\flat G \xrightarrow{i_{G_\flat 1}} (G \rtimes_\alpha H_0)\flat G \xrightarrow{\chi} G$$

and we obtain the same result.
Sketch of the proof (action accessible case).
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As a consequence of condition 3’) the semi-direct product \( G \rtimes_{p^*(\chi)} H \) is isomorphic to the product \( G \times H \):
Sketch of the proof (action accessible case).
As a consequence of condition 3’) the semi-direct product $G \rtimes_{p^*}(\chi) H$ is isomorphic to the product $G \times H$.
Sketch of the proof (action accessible case).
As a consequence of condition 3’) the semi-direct product $G \rtimes_{p^*(\chi)} H$ is isomorphic to the product $G \times H$:

Hence the morphisms $n = (1 \rtimes \partial)j$ and $i_G$ cooperate in $G \rtimes_\alpha H_0$. 

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Sketch of the proof (action accessible case).
As a consequence of condition 3’) the semi-direct product \( G \rtimes_{p^*(\chi)} H \) is isomorphic to the product \( G \times H \):

\[
\begin{array}{c}
\xymatrix{
G \rtimes_{p^*(\chi)} H 
& G \times H \\
\downarrow \rho & \uparrow \pi_G & \downarrow \pi_H \\
G & \langle 1,0 \rangle & H \\
\downarrow i_G & \uparrow \tau & \downarrow \rho & \downarrow j & \downarrow \pi_H \\
G \rtimes_{\alpha} H_0 & G \times H & H \\
\downarrow 1 \rtimes \partial & \xymatrix{ & \downarrow n} & \ar@{-->}[u] \\
& G \rtimes_{\alpha} H_0 & 
\end{array}
\]

Hence the morphisms \( n = (1 \rtimes \partial)j \) and \( i_G \) cooperate in \( G \rtimes_{\alpha} H_0 \). And consequently \([n(H), G] = 0\).
Since the category is action accessible:
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\[ [n(H), G] = 0 \implies [\overline{n(H)}, G] = 0 \]
Since the category is action accessible:

\[ [n(H), G] = 0 \Rightarrow [\overline{n(H)}, G] = 0 \]

and the “Smith is Huq” property holds.
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These conditions allow to construct a split butterfly:
Since the category is action accessible:

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These conditions allow to construct a split butterfly:

which produces a morphism of crossed modules.
Since the category is action accessible:

\[[n(H), G] = 0 \Rightarrow [\overline{n(H)}, G] = 0\]

and the “Smith is Huq” property holds.

These conditions allow to construct a split butterfly:

\[
\begin{array}{ccc}
H & \longrightarrow & \overline{n(H)} \\
\partial & \downarrow & \nabla \\
H_0 & \longrightarrow & H_0 \\
\end{array}
\]

\[
\begin{array}{ccc}
G & \rtimes^\alpha & H_0 \\
\pi_{H_0} & \downarrow & q \\
G \ltimes^H H_0 & \longrightarrow & H_0 \\
\end{array}
\]

which produces a morphism of crossed modules.
By composition we get the required morphism of precrossed modules.
A particular case
Let \( \mathcal{C} \) be semi-abelian.
Let $\mathbb{C}$ be semi-abelian. In the case $p = 1$:

$$\begin{array}{ccc}
H & \xrightarrow{\partial} & H_0 \\
\downarrow & & \downarrow \\
H & & H
\end{array}$$
Let $\mathbb{C}$ be semi-abelian. In the case $p = 1$:

$\begin{array}{ccc}
H & \xrightarrow{\partial} & H_0 \\
\parallel & & \\
H & & 
\end{array}$

1) as before: $(\partial, \xi)$ is a precrossed module;
Let $\mathbb{C}$ be semi-abelian. In the case $p = 1$:

1) as before: $(\partial, \xi)$ is a precrossed module;
2) disappears (equivariance of 1);
Let \( \mathbb{C} \) be semi-abelian. In the case \( p = 1 \):

\[
\begin{array}{c}
\xymatrix{
H \ar[r]^\partial \ar[d] & H_0 \\
H \ar[d] & \\
H 
}
\end{array}
\]

1) as before: \((\partial, \xi)\) is a precrossed module;
2) disappears (equivariance of 1);
3) becomes:

\[
\begin{array}{c}
\xymatrix{
(H \rtimes_\xi H_0)^b H \ar[r]^{\varphi^b 1} \ar[d] & H_0^b H \\
(H \rtimes_\xi H_0)^b H \ar[d]^\xi & H 
}
\end{array}
\]
Let $\mathbb{C}$ be semi-abelian. In the case $p = 1$:

\[
\begin{array}{c}
\begin{array}{ccc}
H & \xrightarrow{\partial} & H_0 \\
\parallel & & \\
H
\end{array}
\end{array}
\]

1) as before: $(\partial, \xi)$ is a precrossed module;
2) disappears (equivariance of 1);
3) becomes:

\[
\begin{array}{c}
\begin{array}{ccc}
(H \rtimes_\xi H_0) \rtimes H & \xrightarrow{\varphi^b 1} & H_0 \rtimes H \\
\parallel & & \\
(H \rtimes_\xi H_0) \rtimes H & \xrightarrow{\chi} & H
\end{array}
\end{array}
\]

and gives a condition for a precrossed module to be a crossed module ("Super-Peiffer").
In the action accessible context, 3) is replaced by 3’) and the previous condition reduces to Peiffer condition.
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Conclusion:

\[
\text{Action accessible} \quad \Rightarrow \quad 3’) \text{ instead of 3)}
\]

\[
\downarrow \quad \Downarrow
\]

\[
\text{“Smith is Huq”} \quad \Leftrightarrow \quad \text{Peiffer} \Rightarrow \text{“Super-Peiffer”}
\]
In the action accessible context, 3) is replaced by 3’) and the previous condition reduces to Peiffer condition.

Conclusion:

Action accessible \( \Rightarrow \) 3’) instead of 3)

\[ \Downarrow \quad \Downarrow \]

“Smith is Huq” \( \Leftrightarrow \) Peiffer \( \Rightarrow \) “Super-Peiffer”

Observe that the implication on the top depends on the property:

\[ [H, K] = 0 \quad \Rightarrow \quad [\overline{H}, K] = 0 \]