

A push forward construction and the comprehensive factorization for internal crossed modules I

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(joint work with S. Mantovani and G. Metere)

Workshop on Category Theory
in honour of George Janelidze, on the occasion of his 60th birthday
Coimbra, July 13, 2012

Introduction

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by means of the pullback along c :

$$\begin{array}{ccccccccc}
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Again, for any group homomorphism $g: G' \rightarrow G$, the pullback construction determines a functor:

$$g^*: \text{OPEXT}(G, A, \phi) \rightarrow \text{OPEXT}(G', A, g^*(\phi)),$$

where $g^*(\phi)$ is given by the composite:

$$\begin{array}{ccc} G' \times A & \overset{g^*(\phi)}{\dashrightarrow} & A \\ & \searrow^{g \times 1} & \nearrow_{\phi} \\ & G \times A & \end{array}$$

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And again a group homomorphism:

$$H_{\phi}^2(G, A) \rightarrow H_{g^*(\phi)}^2(G', A).$$

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$$\phi': G \times A' \dashrightarrow A'$$

and we require that a is equivariant, i.e.:

$$\begin{array}{ccc} G \times A & \xrightarrow{\phi} & A \\ 1 \times a \downarrow & & \downarrow a \\ G \times A' & \xrightarrow{\phi'} & A' \end{array}$$

These data allow to construct the so called *push forward* along a :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{k} & E & \xrightarrow{f} & G \longrightarrow 0 \\
 & & \downarrow a & & \downarrow e & & \parallel \\
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and a group homomorphism:

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Construction of the push forward (for groups):

$$\begin{array}{ccccc}
 & & & & \\
 & & & & \\
 A & \begin{array}{c} \xrightarrow{k} \\ \xrightarrow{a} \end{array} & \begin{array}{c} E \\ A' \end{array} & \begin{array}{c} \xrightarrow{i_E} \\ \xrightarrow{i_{A'}} \end{array} & E \times_{f^*(\phi')} A' \xrightarrow{q} E' \\
 & & & &
 \end{array}$$

where $q = \text{coeq}(i_E k, i_{A'} a)$.

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Universal property:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & A & \xrightarrow{k} & E & \xrightarrow{f} & G & \longrightarrow & 0 \\
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$$\begin{array}{ccccc}
 & k & \rightarrow & E & \xrightarrow{i_E} & E \times_{f^*(\phi')} A' & \xrightarrow{q} & E' \\
 A & \searrow & & & & & & \\
 & a & \rightarrow & A' & \xrightarrow{i_{A'}} & & &
 \end{array}$$

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Universal property:

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 0 & \longrightarrow & A & \xrightarrow{k} & E & \xrightarrow{f} & G & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow p.f. & & \parallel & & \\
 0 & \longrightarrow & A' & \xrightarrow{k'} & E' & \xrightarrow{f'} & G & \longrightarrow & 0 \\
 & & \parallel & & \downarrow & & \parallel & & \\
 0 & \longrightarrow & A' & \xrightarrow{k''} & E'' & \xrightarrow{f''} & G & \longrightarrow & 0
 \end{array}$$

Construction of the push forward (for groups):

$$\begin{array}{ccccc}
 & k & & i_E & \\
 A & \xrightarrow{\quad} & E & \xrightarrow{\quad} & E \times_{f^*(\phi')} A' & \xrightarrow{q} & E' \\
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 \end{array}$$

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- Is there an internal version of this construction?
- Can it be extended to crossed modules?

A push forward construction in semi-abelian categories

Let \mathcal{C} be a semi-abelian category.

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A precrossed module in \mathbb{C} is a morphism

$$\partial: H \rightarrow H_0,$$

together with an internal action

$$\xi: H_0 \triangleright H \rightarrow H,$$

such that the following diagram commutes:

$$\begin{array}{ccc} H_0 \triangleright H & \xrightarrow{\xi} & H \\ \downarrow 1 \triangleright \partial & & \downarrow \partial \\ H_0 \triangleright H_0 & \xrightarrow{\chi} & H_0 \end{array}$$

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If we want ∂ to be a crossed module, we need a further condition, which is not in general the straightforward generalization of the Peiffer condition for crossed modules of groups.

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 However, if \mathbb{C} satisfies the “Smith is Huq” property, the Peiffer condition:

$$\begin{array}{ccc}
 H \triangleright H & \xrightarrow{\chi} & H \\
 \partial b_1 \downarrow & & \parallel \\
 H_0 \triangleright H & \xrightarrow{\xi} & H
 \end{array}$$

turns out to be sufficient to characterize internal crossed modules among precrossed modules (Martins-Ferreira and Van der Linden, '10).

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- He should have presented here a generalization of his result to crossed modules;
- Meanwhile we reinterpreted conditions in terms of internal actions and semi-direct products, obtaining, for push forward of (pre)crossed modules, an equivalent result.

Let \mathbb{C} be a semi-abelian category, ∂ and p two morphisms in \mathbb{C} :

$$\begin{array}{ccc} H & \xrightarrow{\partial} & H_0 \\ p \downarrow & & \\ G & & \end{array}$$

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- 1) there is an action $\xi : H_0 \triangleright H \rightarrow H$ such that (∂, ξ) is a precrossed module;
- 2) there is an action $\alpha : H_0 \triangleright G \rightarrow G$, and p is equivariant:

$$\begin{array}{ccc} H_0 \triangleright H & \xrightarrow{\xi} & H \\ 1 \triangleright p \downarrow & & \downarrow p \\ H_0 \triangleright G & \xrightarrow{\alpha} & G \end{array}$$

3) the following diagram commutes:

$$\begin{array}{ccc}
 (H \times_{\xi} H_0) \wr G & \xrightarrow{\varphi \wr 1} & H_0 \wr G \\
 (p \times 1) \wr 1 \downarrow & & \downarrow \alpha \\
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 \end{array}$$

where φ is defined by the universal property of semi-direct product:

$$\begin{array}{ccccc}
 H & \xrightarrow{i_H} & H \rtimes_{\xi} H_0 & \xleftarrow{i_{H_0}} & H_0 \\
 & \searrow \partial & \downarrow \varphi & \swarrow 1 & \\
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These conditions are sufficient to obtain a push forward construction.

Theorem

There exists an object $G \times^H H_0$, together with a crossed module $\tilde{\partial}: G \rightarrow G \times^H H_0$, with $\text{coker}(\tilde{\partial}) \cong \text{coker}(\partial)$, and a morphism $\tilde{p}_0: H_0 \rightarrow G \times^H H_0$, such that the following diagram is a morphism of precrossed modules:

$$\begin{array}{ccc}
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 & \searrow \partial' & \downarrow p_0 \\
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which is universal in the following sense: for any other morphism (p, p_0) of precrossed modules, where (∂', ξ') is a crossed module and $p_0^*(\xi') = \alpha$,

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 & \searrow \partial' & \downarrow t \\
 & & G_0
 \end{array}$$

p_0 (curved arrow from H_0 to G_0)
 ∂' (curved arrow from G to G_0)

which is universal in the following sense: for any other morphism (p, p_0) of precrossed modules, where (∂', ξ') is a crossed module and $p_0^*(\xi') = \alpha$, there exists a unique factorization t , with $t\tilde{p}_0 = p_0$ and $(1_G, t)$ a morphism of crossed modules.

A remark about the notation.

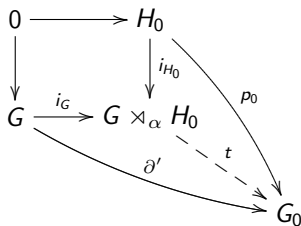
A remark about the notation.

If $H = 0$, conditions 1)–3) reduce to the request of existence of the action α , and the above construction is nothing but semi-direct product:

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The universal property reduces to the universal property of semi-direct product.

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which is in fact weaker:

$$\begin{array}{ccccc} H \wr G & \xrightarrow{i_H \wr 1} & (H \times_{\xi} H_0) \wr G & \xrightarrow{\varphi \wr 1} & H_0 \wr G \\ p \wr 1 \downarrow & & (p \times 1) \wr 1 \downarrow & & \downarrow \alpha \\ G \wr G & \xrightarrow{i_G \wr 1} & (G \times_{\alpha} H_0) \wr G & \xrightarrow{\chi} & G \end{array}$$

and we obtain the same result.

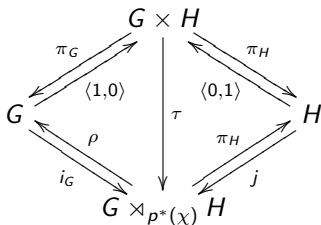
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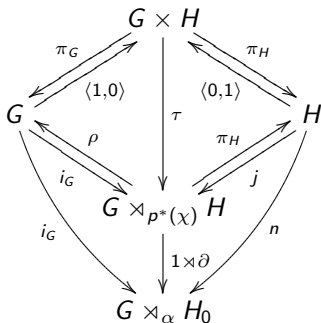
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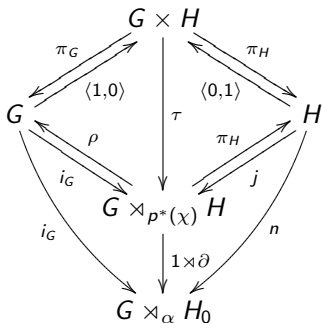
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Hence the morphisms $n = (1 \rtimes \partial)j$ and i_G cooperate in $G \rtimes_{\alpha} H_0$.

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Hence the morphisms $n = (1 \rtimes \partial)j$ and i_G cooperate in $G \rtimes_{\alpha} H_0$.
 And consequently $[n(H), G] = 0$.

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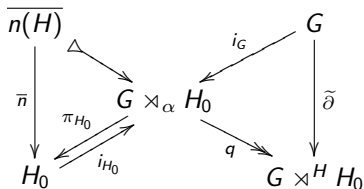
and the “Smith is Huq” property holds.

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These conditions allow to construct a split butterfly:

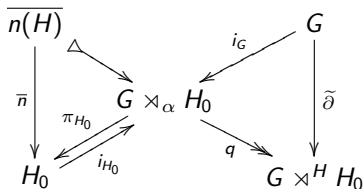


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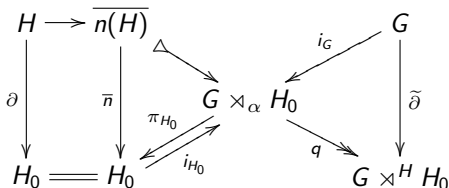
which produces a morphism of crossed modules.

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These conditions allow to construct a split butterfly:



which produces a morphism of crossed modules.

By composition we get the required morphism of precrossed modules.

A particular case

Let \mathbb{C} be semi-abelian.

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- 1) as before: (∂, ξ) is a precrossed module;
- 2) disappears (equivariance of 1);
- 3) becomes:

$$\begin{array}{ccc} (H \rtimes_{\xi} H_0) \wr H & \xrightarrow{\varphi \wr 1} & H_0 \wr H \\ \parallel & & \downarrow \xi \\ (H \rtimes_{\xi} H_0) \wr H & \xrightarrow{\chi} & H \end{array}$$

Let \mathbb{C} be semi-abelian. In the case $p = 1$:

$$\begin{array}{ccc} H & \xrightarrow{\partial} & H_0 \\ \parallel & & \\ H & & \end{array}$$

- 1) as before: (∂, ξ) is a precrossed module;
- 2) disappears (equivariance of 1);
- 3) becomes:

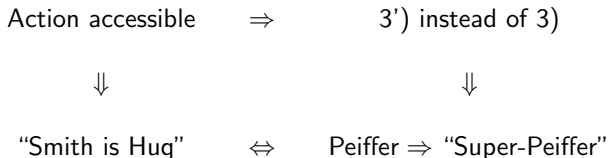
$$\begin{array}{ccc} (H \rtimes_{\xi} H_0) \wr H & \xrightarrow{\varphi \wr 1} & H_0 \wr H \\ \parallel & & \downarrow \xi \\ (H \rtimes_{\xi} H_0) \wr H & \xrightarrow{\chi} & H \end{array}$$

and gives a condition for a precrossed module to be a crossed module (“Super-Peiffer”).

In the action accessible context, 3) is replaced by 3') and the previous condition reduces to Peiffer condition.

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Conclusion:



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Conclusion:

$$\begin{array}{ccc}
 \text{Action accessible} & \Rightarrow & 3') \text{ instead of 3)} \\
 \Downarrow & & \Downarrow \\
 \text{"Smith is Huq"} & \Leftrightarrow & \text{Peiffer} \Rightarrow \text{"Super-Peiffer"}
 \end{array}$$

Observe that the implication on the top depends on the property:

$$[H, K] = 0 \Rightarrow [\overline{H}, K] = 0$$