A push forward construction and the comprehensive factorization for internal crossed modules I

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(joint work with S. Mantovani and G. Metere)

Workshop on Category Theory
in honour of George Janelidze, on the occasion of his 60th birthday
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Introduction

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by means of the pullback along c:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C' \arrow[r] \arrow[d, "c"] & 0 \\
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Again, for any group homomorphism $g\colon G' \to G$, the pullback construction determines a functor:

$$g^*\colon \mathrm{OPEXT}(G, A, \phi) \to \mathrm{OPEXT}(G', A, g^*(\phi))\,,$$

where $g^*(\phi)$ is given by the composite:

$$\begin{tikzcd} G' \times A \arrow[rr, dashed, "g^*(\phi)"] \arrow[dr, "g \times 1"'] & & A \\ & G \times A \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

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And again a group homomorphism:

$$H^2_\phi(G, A) \to H^2_{g^*(\phi)}(G', A) \,.$$

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$$\phi'\colon G \times A' \dashrightarrow A'$$

and we require that a is equivariant, i.e.:

$$\begin{tikzcd}
G \times A \arrow[r, dashed, "\phi"] \arrow[d, "1\times a"'] & A \arrow[d, "a"] \\
G \times A' \arrow[r, dashed, "\phi'"'] & A'
\end{tikzcd}$$

These data allow to construct the so called *push forward* along a:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "k"] \arrow[d, "a"'] \arrow[dr, phantom, "p.f."] & E \arrow[r, "f"] \arrow[d, "e"] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \arrow[r, "k'"'] & E' \arrow[r, "f'"'] & G \arrow[r] & 0 \end{tikzcd}$$

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which determines a functor:

$$a_* : \mathrm{OPEXT}(G, A, \phi) \to \mathrm{OPEXT}(G, A', \phi')$$

and a group homomorphism:

$$H^2_{\phi}(G, A) \to H^2_{\phi'}(G, A').$$

Construction of the push forward (for groups):

$$\begin{tikzcd} & E \arrow[dr, bend left, "i_E"] & & & \\ A \arrow[ur, bend left, "k"] \arrow[dr, bend right, "a"'] & & E \rtimes_{f^*(\phi')} A' \arrow[r, two heads, "q"] & E' \\ & A' \arrow[ur, bend right, "i_{A'}"'] & & \end{tikzcd}$$

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Construction of the push forward (for groups):

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0 \arrow[r] & A' \arrow[r, "k'"'] \arrow[d, equal] & E' \arrow[r, "f'"'] \arrow[d, dashed, "\exists !"'] & G \arrow[r] \arrow[d, equal] & 0 \\
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- Is there an internal version of this construction?
- Can it be extended to crossed modules?

A push forward construction in semi-abelian categories

Let $\mathbb{C}$ be a semi-abelian category.

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A precrossed module in $\mathbb{C}$ is a morphism

$$\partial \colon H \to H_0 \,,$$

together with an internal action

$$\xi \colon H_0 \flat H \to H \,,$$

such that the following diagram commutes:

$$\begin{tikzcd}
H_0 \flat H \arrow[r, "\xi"] \arrow[d, "1\flat\partial"'] & H \arrow[d, "\partial"] \\
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If we want ∂ to be a crossed module, we need a further condition, which is not in general the straightforward generalization of the Peiffer condition for crossed modules of groups.

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However, if $\mathbb{C}$ satisfies the "Smith is Huq" property, the Peiffer condition:

$$\begin{tikzcd}
H \flat H \arrow[r, "\chi"] \arrow[d, "\partial \flat 1"'] & H \arrow[d, equal] \\
H_0 \flat H \arrow[r, "\xi"'] & H
\end{tikzcd}$$

turns out to be sufficient to characterize internal crossed modules among precrossed modules (Martins-Ferreira and Van der Linden, '10).

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- He should have presented here a generalization of his result to crossed modules;
- Meanwhile we reinterpreted conditions in terms of internal actions and semi-direct products, obtaining, for push forward of (pre)crossed modules, an equivalent result.

Let $\mathbb{C}$ be a semi-abelian category, ∂ and p two morphisms in $\mathbb{C}$:

$$\begin{tikzcd} H \arrow[r, "\partial"] \arrow[d, "p"'] & H_0 \\ G & \end{tikzcd}$$

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satisfying the following conditions:

1) there is an action $\xi : H_0 \flat H \to H$ such that (∂, ξ) is a precrossed module;
2) there is an action $\alpha : H_0 \flat G \to G$, and p is equivariant:

$$\begin{tikzcd} H_0 \flat H \arrow[r, "\xi"] \arrow[d, "1 \flat p"'] & H \arrow[d, "p"] \\ H_0 \flat G \arrow[r, "\alpha"'] & G \end{tikzcd}$$

3) the following diagram commutes:

$$\begin{tikzcd}
(H \rtimes_\xi H_0)\flat G \arrow[r, "\varphi\flat 1"] \arrow[d, "(p\rtimes 1)\flat 1"'] & H_0\flat G \arrow[d, "\alpha"] \\
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where φ is defined by the universal property of semi-direct product:

$$\begin{tikzcd}
H \arrow[r, "i_H"] \arrow[dr, "\partial"'] & H \rtimes_{\xi} H_0 \arrow[d, "\varphi"] & H_0 \arrow[l, "i_{H_0}"'] \arrow[dl, "1"] \\
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These conditions are sufficient to obtain a push forward construction.

Theorem

There exists an object $G \rtimes^H H_0$, together with a crossed module $\widetilde{\partial} \colon G \to G \rtimes^H H_0$, with $\mathrm{coker}(\widetilde{\partial}) \cong \mathrm{coker}(\partial)$, and a morphism $\widetilde{p_0} \colon H_0 \to G \rtimes^H H_0$, such that the following diagram is a morphism of precrossed modules:

$$\begin{tikzcd} H \arrow[r, "\partial"] \arrow[d, "p"'] & H_0 \arrow[d, "\widetilde{p_0}"] \\ G \arrow[r, "\widetilde{\partial}"] & G \rtimes^H H_0 \end{tikzcd}$$

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G \arrow[r, "\widetilde{\partial}"] \arrow[drr, bend right, "\partial'"] & G \rtimes^H H_0 & \\
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which is universal in the following sense: for any other morphism (p, p_0) *of precrossed modules, where* (∂', ξ') *is a crossed module and* $p_0^*(\xi') = \alpha$*,*

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G \arrow[r, "\widetilde{\partial}"] \arrow[drr, bend right, "\partial'"] & G \rtimes^H H_0 \arrow[dr, dashed, "t"] & \\
 & & G_0
\end{tikzcd}$$

which is universal in the following sense: for any other morphism (p, p_0) *of precrossed modules, where* (∂', ξ') *is a crossed module and* $p_0^*(\xi') = \alpha$*, there exists a unique factorization* t*, with* $t\widetilde{p_0} = p_0$ *and* $(1_G, t)$ *a morphism of crossed modules.*

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If $H = 0$, conditions 1)–3) reduce to the request of existence of the action α, and the above construction is nothing but semi-direct product:

$$\begin{tikzcd}
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The universal property reduces to the universal property of semi-direct product.

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3′)
$$\begin{tikzcd} H\flat G \arrow[r, "\partial\flat 1"] \arrow[d, "p\flat 1"'] & H_0\flat G \arrow[d, "\alpha"] \\ G\flat G \arrow[r, "\chi"'] & G \end{tikzcd}$$

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which is in fact weaker:

$$\begin{tikzcd}
H\flat G \arrow[r, "i_H\flat 1"] \arrow[d, "p\flat 1"'] & (H \rtimes_\xi H_0)\flat G \arrow[r, "\varphi\flat 1"] \arrow[d, "(p\rtimes 1)\flat 1"'] & H_0\flat G \arrow[d, "\alpha"] \\
G\flat G \arrow[r, "i_G\flat 1"'] & (G \rtimes_\alpha H_0)\flat G \arrow[r, "\chi"'] & G
\end{tikzcd}$$

and we obtain the same result.

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G \arrow[ur, shift left, "\langle 1{,}0 \rangle"'] \arrow[dr, shift right, "i_G"'] & & H \arrow[ul, shift left, "\langle 0{,}1 \rangle"] \arrow[dl, shift left, "j"] \\
& G \rtimes_{p^*(\chi)} H \arrow[ul, shift right, "\rho"'] \arrow[ur, shift left, "\pi_H"] &
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 & & G \rtimes_{p^*(\chi)} H \arrow[ull, shift right, "\rho"'] \arrow[urr, shift left, "\pi_H"] \arrow[d, "1 \rtimes \partial"] & & \\
 & & G \rtimes_{\alpha} H_0 & &
\end{tikzcd}
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Hence the morphisms $n = (1 \rtimes \partial)j$ and i_G cooperate in $G \rtimes_{\alpha} H_0$.

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G \arrow[ur, shift left, "\langle 1{,}0\rangle"'] \arrow[dr, shift left, "i_G"'] \arrow[ddr, bend right, "i_G"'] & & H \arrow[ul, shift left, "\langle 0{,}1\rangle"] \arrow[dl, shift left, "j"] \arrow[ddl, bend left, "n"] \\
 & G \rtimes_{p^*(\chi)} H \arrow[ul, shift left, "\rho"'] \arrow[ur, shift left, "\pi_H"] \arrow[d, "1 \rtimes \partial"] & \\
 & G \rtimes_{\alpha} H_0 &
\end{tikzcd}$$

Hence the morphisms $n = (1 \rtimes \partial)j$ and i_G cooperate in $G \rtimes_\alpha H_0$.
And consequently $[n(H), G] = 0$.

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These conditions allow to construct a split butterfly:

$$\begin{tikzcd}
\overline{n(H)} \arrow[dd, "\overline{n}"'] \arrow[dr, tail] & & G \arrow[dl, "i_G"'] \arrow[dd, "\widetilde{\partial}"] \\
& G \rtimes_\alpha H_0 \arrow[dl, shift right, "\pi_{H_0}"'] \arrow[dr, two heads, "q"] & \\
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H \arrow[r] \arrow[dd, "\partial"'] & \overline{n(H)} \arrow[dd, "\overline{n}"'] \arrow[dr, tail] & & G \arrow[dl, "i_G"'] \arrow[dd, "\widetilde{\partial}"] \\
& & G \rtimes_\alpha H_0 \arrow[dl, shift right, "\pi_{H_0}"'] \arrow[dr, two heads, "q"'] & \\
H_0 \arrow[r, equal] & H_0 \arrow[ur, shift right, "i_{H_0}"'] & & G \rtimes^H H_0
\end{tikzcd}$$

which produces a morphism of crossed modules.
By composition we get the required morphism of precrossed modules.

A particular case

Let $\mathbb{C}$ be semi-abelian.

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3) becomes:

$$\begin{tikzcd} (H \rtimes_\xi H_0)\flat H \arrow[r, "\varphi\flat 1"] \arrow[d, equal] & H_0\flat H \arrow[d, "\xi"] \\ (H \rtimes_\xi H_0)\flat H \arrow[r, "\chi"'] & H \end{tikzcd}$$

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and gives a condition for a precrossed module to be a crossed module ("Super-Peiffer").

In the action accessible context, 3) is replaced by 3') and the previous condition reduces to Peiffer condition.

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Conclusion:

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\begin{array}{ccc}
\text{Action accessible} & \Rightarrow & \text{3') instead of 3)} \\
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\text{“Smith is Huq”} & \Leftrightarrow & \text{Peiffer} \Rightarrow \text{“Super-Peiffer”}
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In the action accessible context, 3) is replaced by 3') and the previous condition reduces to Peiffer condition.
Conclusion:

$$\begin{array}{ccc} \text{Action accessible} & \Rightarrow & \text{3') instead of 3)} \\ \Downarrow & & \Downarrow \\ \text{“Smith is Huq”} & \Leftrightarrow & \text{Peiffer} \Rightarrow \text{“Super-Peiffer”} \end{array}$$

Observe that the implication on the top depends on the property:

$$[H, K] = 0 \quad \Rightarrow \quad \left[\overline{H}, K\right] = 0$$