Products in the category of approach spaces as models for complexity

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Workshop on Category Theory, Coimbra, July 2012
Conference in honor of George Janelidze, on the occasion of his 60-th birthday

Complexity of algorithms

Complexity of certain types of algorithms, is described as a solution of some recurrence equation. For Mergesort the running time $f : \mathbb{N} \to]0, \infty]$ is a solution of the equation:

$$\begin{cases} f(n) = c \quad \textit{for} \quad n = 1 \\ f(n) = a.f[\frac{n}{b}] + h(n) \quad \textit{whenever } n \neq 1 \end{cases}$$

for given a, b, c *and* $h : \mathbb{N} \to]0, \infty]$.

M. P. Schellekens, The Smyth completion: A common foundation for denotational semantics and complexity analysis, Elect. Notes Theoret. Comp. Sci., (1995).

Complexity of algorithms 2

Calculations of running time of other examples like Quicksort fit into the following recurrence equation:

$$\begin{cases} f(n) = c_n \quad \text{for} \quad 1 \leq n \leq k \\ f(n) = \Sigma_{i=1}^{i=k} a_i . f(n-i) + h(n) \quad \text{whenever } n > k \end{cases}$$

for given k, a_i *and* $h : \mathbb{N} \to]0, \infty]$

S. Romaguera and O. Valero, A common Mathematical Framework for Asymptotic Complexity Analysis and Denotational Semantics for Recursive Programs Based on Complexity spaces, International Journal of Computer Mathematics, 2012.

Associated fixed point problem

Reformulating the problem as a fixed point result:
$X =]0, \infty]^{\mathbb{N}}$ and $\Phi : X \to X : g \mapsto \Phi g$.

$$\begin{cases} \Phi g(n) = c_n \quad \textit{for} \quad 1 \leq n \leq k \\ \Phi g(n) = \Sigma_{i=1}^{i=k} a_i . g(n-i) + h(n) \quad \textit{whenever } n > k \end{cases}$$

Other references

- S. Romaguera, M.P. Schellekens, P. Tirado, O. Valero, Contraction selfmaps on complexity spaces and ExpoDC algorithms, Amer. Inst. Physics Proceedings, (2007).

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- L. M. García-Raffi, S. Romaguera, M. P. Schellekens, Applications of the complexity space to the general probabilistic divide and conquer algorithms, J. Math. Anal. Appl. (2008).

Solutions of the fixed point problems

Method

- The complexity distance.

$$d_{\mathcal{C}}(f,g) = \Sigma_{n\in\mathbb{N}} \frac{1}{2^n}.[(\frac{1}{g(n)} - \frac{1}{f(n)}) \vee 0]$$

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- Restrict to $\Phi : \mathcal{C} \to \mathcal{C}$, $d_{\mathcal{C}}$-Lipschitz with factor strictly smaller than 1
- Apply the Banach fixed point theorem for quasi metric spaces to obtain a unique fixed point for $\Phi : \mathcal{C} \to \mathcal{C}$.

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- Changing the categorical context $\mapsto$ develop a method applicable to a larger class of recursive algorithms, containing all the previous examples.
- Construct App of approach spaces and contractions as morphisms.
- Categorical product in App $\mapsto$ complexity approach space $]0, \infty]^{\mathbb{N}}$, compatibility with the product in Top and with the pointwise order.

Approach spaces 1

From Top to App: Convergence in an approach space X is described by means of a limit operator.

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Objects (X, λ) *with* $\lambda : FX \to [0, \infty]^X : \mathcal{F} \mapsto \lambda\mathcal{F}$ *satisfying suitable axioms.*
A map $f : (X, \lambda_X) \to (Y, \lambda_Y)$ *is a contraction if*

$$\lambda_Y(\text{stack} f(\mathcal{F})) \circ f \leq \lambda_X \mathcal{F}$$

for every $\mathcal{F} \in F(X)$.

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Top $\to$ App *is a concretely coreflective full embedding.*
Given $X = (X, \lambda)$ *we denote its topological coreflection as* $(X, \mathcal{T}_X)$, *with* $\mathcal{T}_X$ *defined by*

$$\mathcal{F} \to x \Leftrightarrow \lambda\mathcal{F}(x) = 0.$$

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Objects are $(X, \mathcal{G})$ *with* $\mathcal{G}$ *an ideal of quasi metrics* $d : X \times X \to [0, \infty]$ *satisfying a certain saturation condition.*
A map $f : (X, \mathcal{G}_X) \to (Y, \mathcal{G}_Y)$ *is a contraction if*

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qMet $\to$ App *is a concretely coreflective full embedding.*
Given $X = (X, \mathcal{G})$ we denote its quasi metric coreflection as (X, d_X), with d_X defined as

$$d_X = \sup \mathcal{G}.$$

The category of Approach spaces 1

The two constructs are concretely isomorphic. The transition from gauges to limit operators:

For an approach space with gauge $\mathcal{G}$, for a filter $\mathcal{F}$ and $x \in X$ the limit operator is:

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In particular: for a quasi metric space (X,q) with gauge $\{d \mid d \leq q\}$ the limit operator of a sequence $(x_n)_n$ is:

$$\lambda(x_n)_n(x) = limsup_{n\to\infty}\ q(x,x_n)$$

The category of Approach spaces 2

App is a topological construct:

Structured source $(f_i : X \to (X_i, \lambda_i))_{i \in I}$, *initial limit operator on* $\mathcal{F} \in FX$*:*

$$\lambda \mathcal{F} = \sup_{i \in I} \lambda_i(stackf_i(\mathcal{F})) \circ f_i$$

Approach complexity spaces 1

Structure $X = Z^Y$:

- $Z = (]0, \infty], \leq)$ or $Z = ([0, \infty], \leq)$, dcpo for the usual order.
- p the quasi metric defined by

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  induces the Scott topology $\sigma(Z, \leq)$.
- For Y arbitrary, $X = Z^Y$ is a dcpo for the pointwise order $\leq$.
- Endow X with the product in App, i.e. the initial lift of the source

$$(pr_y : X \to (Z, p))_{y \in Y}.$$

  The space $X = (X, \lambda)$ is called the complexity approach space.

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$$= \sup_{y \in Y} \limsup_{k \to \infty} \left(\frac{1}{g_k(y)} - \frac{1}{f(y)}\right) \vee 0$$

$$\lambda(g_k)_k(f) \leq \alpha \Leftrightarrow$$

$$\forall y \in Y, \forall \eta > 0, \exists j_y, \forall k \geq j_y \quad \left(\frac{1}{g_k(y)} - \frac{1}{f(y)}\right) \vee 0 < \alpha + \eta$$

Fixed points 1

$X = Z^Y$, suppose Y is endowed with a binary irreflexive relation $\prec$ and for $y \in Y$ let

$$Y_y = \{u \in Y \mid u \prec y\}$$

the initial segment of y. We assume $(Y, \prec)$ is well founded.

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Examples for Y:

- $(\mathbb{N}, <)$ where $<$ is the strict relation associated to the usual well order
- finite powers $\mathbb{N}^n$, with the "strict pointwise relation" $(n_i)_i \prec (m_i)_i$ if and only if $n_i < m_i$ for $i = 1, \cdots, n$.

Fixed points 2

$X = Z^Y$ is the complexity approach space, and $\Phi : X \to X$ is of the following type

- *there exists $h \in X$ taking only finite values*
- *for every $y \in Y$, not minimal, there exists $\Psi_y : Z^{Y_y} \to Z$, such that Φ satisfies*

$$\Phi g(y) = \begin{cases} h(y) + \Psi_y((g(u))_{u \in Y_y}) & y \text{ not minimal} \\ h(y) & y \text{ minimal} \end{cases}$$

for $g \in X$.

Sufficient conditions

For $y \in Y$, y not minimal Z^{Y_y} has pointwise order and addition inherited from Z.
For $a \in Z^{Y_y}$ with $a_u = s$ for every $u \in Y_y$ we write $(a_u)_u = \underline{s}$.

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$$\Psi_y : Z^{Y_y} \to Z$$

satisfies the following conditions:

1. *Monotone:* $a \leq b$ *in* $Z^{Y_y} \Rightarrow \Psi_y(a) \leq \Psi_y(b)$
2. *Subadditive:* $\Psi_y(a+b) \leq \Psi_y(a) + \Psi_y(b)$
3. *Limit:* $\forall \varepsilon > 0\ \exists \delta > 0 : \Psi_y(\underline{s}) \leq \varepsilon$ *whenever* $s \leq \delta$
4. *Finiteness: If* $a \in Z^{Y_y}$ *has only finite coordinates then* $\Psi_y(a)$ *is finite.*

Results

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- *f takes finite values*
- *f is (the unique) fixed point of Φ*

Finite values

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Sketch of Proof: if f takes infinite values on Y, let

$$U = \{u \in Y \mid f(u) = \infty\}$$

Since $U \neq \emptyset$ there exists a minimal element y in $(U, \prec)$. In particular $f(y) = \infty$. By finiteness of $h(y)$ we have y is not minimal in Y.
$f(y) \leq \Phi f(y)$ so we have $\Phi f(y) = \infty$.
But

$$\Phi f(y) = h(y) + \Psi_y((f(u))_{u \in Y_y}).$$

In view of the minimality of y in U we have $f(u) < \infty$ for $u \in Y_y$, so by the finiteness of Ψ_y a contradiction follows.

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We use the pointwise expression

$$\lambda(\Phi^n h)_n(f) = \sup_{u \in Y} \limsup_{n \to \infty} p(f(u), \Phi^n h(u))$$

of the complexity approach space.

For $u \in Y_y$ we take j_u such that for $k \geq j_u$

$$p(f(u), \Phi^k h(u))) \leq \frac{\delta}{f(u).(f(u) - \delta)}$$

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$$\leq \Psi_y(\underline{\delta}) + \Phi^{k+1} h(y) \leq \Psi_y(\underline{\delta}) + f(y)$$

$$\Phi f(y) \leq \varepsilon + f(y).$$

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- Several variables as inputs (ExpoDC algorithm): $Y = \mathbb{N}_0 \times \mathbb{N}_0$ and let $Z = [0, \infty]$. Φ on $X = Z^Y$: for $g \in X$

$$\Phi g(m,n) = \begin{cases} 0 & \text{if } n = 1 \\ g(m, \frac{n}{2}) + M(\frac{mn}{2}, \frac{mn}{2}) & \text{if } n \text{ is even} \\ g(m, n-1) + M(m, (n-1)m) & \text{otherwise} \end{cases}$$

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Binary relation on Y:

$$(m,n) \prec (m',n') \Leftrightarrow m = m' \text{ and } n < n'.$$

For $y = (m, n)$ not minimal, $\Psi_y : [0, \infty]^{Y_y} \to [0, \infty]$ is defined by

$$\Psi_y(a) = \begin{cases} a_{m,\frac{n}{2}} & \text{if } n \text{ is even} \\ a_{m,n-1} & \text{otherwise} \end{cases}$$

with $a = (a_{m,1}, \cdots, a_{m,n-1})$.

Examples 2

- The vertex covering problem
  Given a graph $G = (V, E)$, does there exist a subset $W \subseteq V$ of the set of vertices of G, of size k, such that for every edge $(u, v) \in E$ of G, either $u \in W$ or $v \in W$?

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  Inputs (n, k), where n is the number of all vertices of the graph and k is the size of the subset W:
  Take
  $$Y = \{(n, k) \in \mathbb{N} \times \mathbb{N} \mid k \leq n\}$$
  endowed with the strict pointwise relation,
  $Z = [0, \infty]$ and $h = pr_1 : Y \to Z$ the first projection.

$\Phi : Z^Y \to Z^Y$ is

$$\Phi(g)(n,k) = \begin{cases} h(n,k) + 2g(n-1,k-1) & (n,k) \text{ not minimal} \\ h(n,k) & \text{otherwise.} \end{cases}$$

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For (n,k) not minimal $\Psi_{(n,k)} : Z^{Y_{(n,k)}} \to Z$ is defined as

$$\Psi_{(n,k)}(a) = 2a_{(n-1,k-1)}.$$

Upper- and lowerbounds

If Φ is of the same type and $g \in X$ is such that

$$\Phi(g) \leq g$$

(resp $g \leq \Phi g$) then the fixed point f satisfies

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(resp $g \leq f$*).*

For y not minimal. Suppose $f(u) \leq g(u)$ for all $u \in Y_y$. The result follows using monotonicity of Ψ_y and the fact that

$$f(y) = \Phi f(y) = h(y) + \Psi_y((f(u))_{u \in Y_y}) \leq$$
$$h(y) + \Psi_y((g(u))_{u \in Y_y}) = \Phi g(y) \leq g(y).$$

Comparison

$Z = (]0,\infty], \leq)$ with $p(x,y) = (\frac{1}{y} - \frac{1}{x}) \vee 0$.

For $Y = \mathbb{N}$ and $X = (]0,\infty]^{\mathbb{N}}, \leq)$ and $(g_k)_k$ and f in X.

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Complexity quasi metric structure:

$$d_{\mathcal{C}}(f, g_k) = \Sigma_{n\in\mathbb{N}} \frac{1}{2^n}.[(\frac{1}{g_k(n)} - \frac{1}{f(n)}) \vee 0]$$

$$= \Sigma_{n\in\mathbb{N}} \frac{1}{2^n}.p(f(n), g_k(n))$$

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- $(\mathcal{C}, d_{\mathcal{C}})$ is not compatible with the trace of the producttopology.

complexity approach structure: Categorical product on $X =]0, \infty]^{\mathbb{N}}$

- The gauge on X is the saturation of the ideal generated by the collection

$$\{p \circ pr_n \times pr_n \mid n \in \mathbb{N}\}$$

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$$\lambda(g_k)_k(f) = \sup_{n \in \mathbb{N}} \limsup_{k \to \infty} (\frac{1}{g_k(n)} - \frac{1}{f(n)}) \vee 0$$

- $$\lambda(g_k)_k(f) \leq \alpha \Leftrightarrow$$

$$\forall \eta > 0, \forall n \in \mathbb{N}, \exists j_n, \forall k \geq j_n \ (\frac{1}{g_k(n)} - \frac{1}{f(n)}) \vee 0 < \alpha + \eta$$