WorkCT12

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Reflections into idempotent subvarieties of universal algebras and their Galois theories

Isabel Xarez

PhD student at the University of Aveiro

Categorical version of monotone-light factorization for continuous maps of compact Hausdorff spaces was obtained in **“On Localization and Stabilization for Factorization Systems”**, A. Carboni, G. Janelidze, G. M. Kelly, R. Paré, 1997.

The results on the reflection of semigroups into semillatices obtained in **“Limit preservation properties of the greatest semilattice image functor”**, G. Janelidze, V. Lann, L. Màrki, 2008, look similar to the results on the reflection of compact Hausdorff spaces into Stone spaces.

In **“Admissibility, Stable units and connected components”** , J.J.Xarez, 2011, it is shown that this is not only similarity, but two special cases of the same ’theory’.

My work begins by applying this to (semigroups again and) universal algebras.

1 Preservation of finite products

In **"Limit preservation properties of the greatest semilattice image functor"**, G. Janelidze, V. Lann, L. Màrki, 2008, it is shown that the reflector $D : \mathbf{SGr} \to \mathbf{SLat}$ preserves finite products.

How did they prove this?

Consider the reflection $H \vdash B : \mathbf{SGr} \to \mathbf{Band}$. They noticed first that $B(\mathbb{N} \times \mathbb{N}) = 1$ which implies that the map $\gamma_r : Q \to HB(Q \times R);\ q \mapsto [(q, r)]$ is, actually, a homomorphism, for every fixed $r \in R$. Hence, it induces a homomorphism $D(Q) \to D(Q \times R)$.

Now, notice that $\mathbb{N}$ is just the free semigroup on one generator. All this can, then, be generalized as follows:

In fact, for a reflection $H \vdash D : \mathcal{A} \to \mathcal{B}$ from a finitely complete category $\mathcal{A}$ into a full subcategory $\mathcal{B}$, subject to the following data **I**:

(1) there exists a functor $U : \mathcal{A} \to \mathbf{Set}$ which preserves finite limits and reflects isomorphisms ;

(2) every map $U(\eta_A)$ is a surjection, for every unit morphism η_A, $A \in \mathcal{A}$.

D preserves the product $Q \times R$ provided for all $q \in U(Q)$ and $r \in U(R)$, there exist morphisms $\gamma_r : Q \to HD(Q \times R)$ and $\gamma_q : R \to HD(Q \times R)$, such that

$$U(\gamma_r)(a) = U(\eta_{Q\times R})(a, r),$$

for all $a \in U(Q)$, with r fixed.

$$U(\gamma_q)(b) = U(\eta_{Q\times R})(q, b),$$

for all $b \in U(R)$, with q fixed.

Further conclusions follow from this fact:

(1) Let $H \vdash D : \mathcal{A} \to \mathcal{B}$ be a reflection from a variety of universal algebras $\mathcal{A}$ into an idempotent subvariety $\mathcal{B}$[a].

$$D \text{ preserves finite products}$$
$$\text{if and only if}$$
$$D(F(x) \times F(x)) \cong \mathbf{1},\ ^{\mathrm{b}}$$

Then, J. Xarez suggested to use its Data in the paper above in order to find out if the scope of this work could be enlarged.

(2) Under data **I**, if $U_{T;A} : \mathcal{A}(T, A) \to \mathbf{Set}(\{*\}, U(A))$[c] is a surjection for every object $A \in \mathcal{A}$, with T a terminal object in $\mathcal{A}$ then D preserves finite products.

[a] every x in any $X \in \mathcal{B}$ is a subalgebra

[b] $F(x)$ is the free algebra on one generator

[c] in varieties of universal algebras this is equivalent to $\mathcal{A}$ being idempotent

(3) It follows either from (1) or from (2) that finite products are always preserved if not only $\mathcal{B}$ but also the variety $\mathcal{A}$ is idempotent.

Since the reflections above preserve finite products, they have stable units if and only if they are semi-left-exact, as follows from: "**Admissibility**, **Stable units and connected components**", J.J.Xarez, 2011.

2 The prefactorization system $(\mathcal{E}_D, \mathcal{M}_D)$ derived from reflective subvarieties

$\mathcal{E}_D$ is the class of homomorphisms $e : S \to L$ in $\mathcal{C}$ such that:

- $[l]_{\sim_L} \cap \; e(S) \neq \emptyset$,
- $e(s) \sim_L e(s') \Rightarrow s \sim_S s'$,

for every $s, s' \in S$ and $l \in L$.

If the reflection is simple, then this prefactorization system is a factorization system and $\mathcal{M}_D$ is the class of homomorphisms $m : S \to L$ in $\mathcal{C}$ such that

$$m_{|_{[s]_{\sim_S}}} : [s]_{\sim_S} \to [m(s)]_{\sim_L}$$

is an isomorphism, for every $s \in S$.

3 Simple = Semi-left-exact

If the unit morphisms $\eta_S : S \to HD(S)$ of a reflection $H \vdash D : \mathcal{A} \to \mathcal{B}$ from a finitely complete category into a full subcategory are effective descent morphisms in $\mathcal{A}$, then the reflection is simple if and only if it is semi-left-exact.

(This follows from a fact proved in the first paper, namely: If in a pullback constituted by two commutative squares the left square is a pullback whose bottom arrow is an effective descent morphism, then the right square is a pullback too.)

That is the case of varieties of universal algebras, since the unit morphisms of any reflection into a subvariety are surjective homomorphisms, which are just the effective descent morphisms in any variety of universal algebras.

4 Galois groupoid = equivalence relation

In both reflections $D : \mathbf{Band} \to \mathbf{SLat}$ and $D : \mathbf{CommSgr} \to \mathbf{SLat}$ the following property holds for every effective descent morphism $p : A \to B$:

$$b \sim_B b' \Rightarrow \ \exists\, a, a' \in A, \ \text{with} \ \ a \sim_A a', \ \ p(a) \in \langle b \rangle_B, \ \ p(a') \in \langle b' \rangle_B, \tag{1}$$

for all $b, b' \in B$.[a]

[a] $\langle b \rangle_B$ denotes the subalgebra of B generated by b

Let $H \vdash D : \mathcal{C} \to \mathcal{X}$ be a simple (= semi-left-exact) reflection into an idempotent subvariety $\mathcal{X}$ which satisfies the property (1), for every effective descent morphisms in $\mathcal{C}$:

If $\sigma : A \to B$ is an effective descent morphism in $\mathcal{C}$ and $\pi_1 \in \mathcal{M}_D$ in the pullback below, then the following conditions (i) and (ii) are equivalent:

(i) In the following pullback $D(\pi_1)$ and $D(\pi_2)$ are jointly-monic;

(ii) the reflector D preserves this pullback.

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "f"] \\ A \arrow[r, "\sigma"] & B \end{tikzcd}$$

Under these conditions (i)⇔(ii),

- the Galois groupoid $\mathrm{Gal}(L, \sigma)$ of a Galois descent homomorphism $\sigma : A \rightarrow B$ is the equivalence relation given by the kernel pair of $D(\sigma)$, in $\mathcal{X}$;
- $\mathcal{M}^*{}_D/B = \mathcal{M}_D/B$.

For instance:

σ is any Galois descent homomorphism, in the reflection $D : \mathbf{Band} \rightarrow \mathbf{SLat}$;

σ is a Galois descent homomorphism and B has cancellation, or B is finitely generated, or each of its archimedean classes has one idempotent, in the reflection $D : \mathbf{CommSgr} \rightarrow \mathbf{SLat}$.

These results were also generalized for semi-left-exact reflections $H \vdash D : \mathcal{A} \rightarrow \mathcal{B}$ from a finitely complete category $\mathcal{A}$ into a full subcategory $\mathcal{B}$, under data I .

5 The class $\mathcal{E}'_D$ of stably-vertical morphisms

- Let $H \vdash D : \mathcal{C} \to \mathcal{X}$ be a reflection into a subvariety of universal algebras;
- let $\langle x \rangle_C$ denote the subalgebra of $C \in \mathcal{C}$, generated by $x \in C$;
- let $\mathcal{F}$ denote the class of homomorphisms $f : S \to L$ in $\mathcal{C}$, such that $\langle l \rangle_L \cap f(S) \neq \emptyset$.

$$\mathcal{E}'_D \subseteq \mathcal{F},$$

for any reflection into a subvariety of universal algebras.

5.1 $\mathcal{X}$ idempotent

If $\mathcal{X}$ is idempotent, then the following conditions (a) and (b) are equivalent:

(a) For all the pullback diagrams in $\mathcal{C}$, such that $g \in \mathcal{E}_D \cap \mathcal{F}$,

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r] & C \end{tikzcd}$$

$D(\pi_1)$ and $D(\pi_2)$ are jointly-monic;

(b) $\mathcal{E}'_D = \mathcal{E}_D \cap \mathcal{F}$.

This result characterizes the class of stably-vertical morphisms in the reflection $D : \mathbf{Band} \to \mathbf{SLat}$.

Under these equivalent conditions the reflection $D : \mathcal{C} \to \mathcal{X}$ with $\mathcal{X}$ idempotent has stable units, since $\eta_C \in \mathcal{E}_D \cap \mathcal{F}$.

This result was also generalized for a reflection $H \vdash D : \mathcal{A} \to \mathcal{B}$ from a finitely complete category $\mathcal{A}$ into a full subcategory $\mathcal{B}$, subject to <u>data $\mathbf{I}$</u>, provided $U_{T;A} : \mathcal{A}(T, A) \to \mathbf{Set}(\{*\}, U(A))$ is a surjection for every object $A \in \mathcal{A}$, with T a terminal object in $\mathcal{A}$.

In the reflection **CommSgr** $\rightarrow$ **SLat** things were not so easy and, then, G. Janelidze suggested to try free semigroups. From this suggestion followed the next facts.

Consider again a reflection $H \vdash D : \mathcal{C} \rightarrow \mathcal{X}$ into a subvariety of universal algebras and the free adjunction $\langle F, U, \delta, \varepsilon \rangle : \mathbf{Set} \rightarrow \mathcal{C}$.[a]

A homomorphism $e : S \rightarrow L$ belongs to $\mathcal{E}'_D$ only if its pullback $\varepsilon_L^*(e)$ along ε_L, belongs to $\mathcal{F}$.

If the reflection is into an idempotent subvariety and $\varepsilon_A : FU(A) \rightarrow A$ satisfies property (1), [b] for every $A \in \mathcal{C}$, then the following two conditions are equivalent:

[a] $\varepsilon_A : FU(A) \rightarrow A$ is an effective descent morphism, for all $A \in \mathcal{C}$.

[b] $b \sim_B b' \Rightarrow \exists\ a, a' \in A$, with $a \sim_A a'$, $p(a) \in \langle b \rangle_B$, $p(a') \in \langle b' \rangle_B$

(i) For all the diagrams in $\mathcal{C}$, where both squares are pullbacks, such that $\varepsilon_L^*(e) \in \mathcal{E}_D \cap \mathcal{F}$,

$$\begin{tikzcd}
{} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & {} \arrow[r] \arrow[d, "\varepsilon_L^*(e)"] & S \arrow[d, "e"] \\
FU(A) \arrow[r, "FU(f)"'] & FU(L) \arrow[r, "\varepsilon_L"'] & L
\end{tikzcd}$$

$HD(\pi_1)$ and $HD(\pi_2)$ are jointly-monic.

(ii) A homomorphism $e : S \to L$ belongs to $\mathcal{E}'_D$ if and only if $\varepsilon_L^*(e) \in \mathcal{E}_D \cap \mathcal{F}$.

This result characterizes the class $\mathcal{E}'{}_D$ in the reflection **CommSgr** $\to$ **SLat**.

This result was also generalized for a reflection $H \vdash D : \mathcal{A} \to \mathcal{B}$ from a finitely complete category $\mathcal{A}$ into a full subcategory $\mathcal{B}$, subject to data **I**, provided $U(\varepsilon_A)$ and $U_{T;A} : \mathcal{A}(T, A) \to \mathbf{Set}(\{*\}, U(A))$ are surjections for every object $A \in \mathcal{A}$, with T a terminal object in $\mathcal{A}$

6 Separable, purely inseparable and normal morphisms

Consider a reflection $H \vdash D : \mathcal{C} \to \mathcal{X}$ into a subvariety of universal algebras. If $D(u)$ and $D(v)$ are jointly-monic, for a kernel pair (u, v) of a homomorphism α, then:

- $\alpha : A \to B$ is separable if and only if

$$Ker(\alpha) \cap \sim_A = \Delta$$

- $\alpha : A \to B$ is purely inseparable if and only if

$$Ker(\alpha) \subseteq \sim_A$$

- $\alpha : A \to B$ is normal if and only if the next two conditions hold:

  1. $\sim_A \circ Ker(\alpha) \subseteq Ker(\alpha) \circ \sim_A$
  2. $Ker(\alpha) \cap \sim_A = \Delta$

[a]

For instance, these characterizations hold

for all the homomorphisms in the reflection $D : \mathbf{Band} \to \mathbf{SLat}$ (in this reflection normal homomorphisms were already characterized by V. Lann);

and, in the reflection $D : \mathbf{CommSgr} \to \mathbf{SLat}$, for all the homomorphisms whose codomain has cancellation law, or is finitely generated, or each of its congruence classes has an idempotent.

[a] Δ denotes the equality relation, $Ker(\alpha)$ denotes the kernel pair of α and $\sim_A$ denotes the congruence on A induced by the reflection

6.1 Factorizations in Band → SLat

In the reflection of bands into semilattices ${\mathcal{E}'}_D = \mathcal{E}_D \cap \mathcal{E}$, where $\mathcal{E} = \{surjective\ homomorphisms\}$ then there is an (Ins, Sep) factorization system, with $Ins = \mathcal{E}_D \cap \mathcal{E}$.

On the other hand there is no monotone-light factorization, $({\mathcal{E}'}_D, {\mathcal{M}^*}_D)$, since monomorphisms clearly belong to the class of separable morphisms, while there are monomorphisms that do not belong to the class $\mathcal{M}_D = {\mathcal{M}^*}_D$.