The partial real numbers and the order completion of function rings in pointfree topology

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Joint work with J. Gutiérrez García and J. Picado

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Outline

1. Introduction
   - Preliminaries
   - Pointfree Topology
2. Dedekind completion
   - The Frame of Partial Reals
   - The Dedekind Order Completion of $C(L)$

Theorem (Banaschewski and Hong, 2003)

Let L be a completely regular frame. Then, the following are equivalent:

- $\mathrm{C}(L)$ *is order complete;*
- *L is extremally disconnected.*

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For any subset A of a partially ordered set $(P, \leq)$ we denote by $\bigvee^P A$ (resp. $\bigwedge^P A$) the supremum (resp. infimum) of A in P in case it exists (we shall omit the superscript if it is clear from the context).

Definition

A partially ordered set $(P, \leq)$ is called *Dedekind order complete* if every non–void subset A of P which is bounded from above has a supremum and, dually, every non–void subset B of P which is bounded from below has a infimum.

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Definition (Dedekind order completion)

A *(Dedekind) order completion* of a poset $(P, \leq)$ is a pair $(P^{\#}, \Phi)$ where

- $P^{\#}$ is a Dedekind order complete poset,
- $\Phi\colon P \to P^{\#}$ is an order embedding (usually $P \subseteq P^{\#}$) that preserves all suprema and infima that exists in P and satisfies

$$\hat{p} = \bigvee^{P^{\#}} \{\Phi(p) \in \Phi(P) \mid \Phi(p) \leq \hat{p}\}$$
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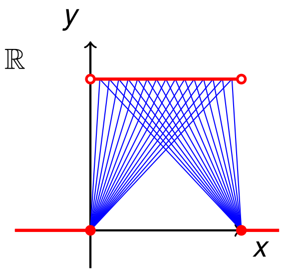

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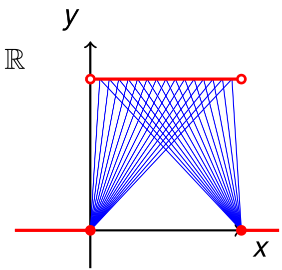

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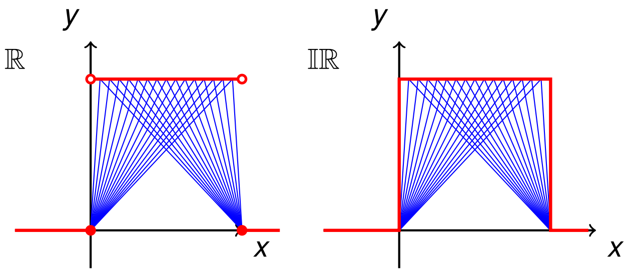

Partial real line

Let $\mathbb{IR}$ denote the set of compact intervals $\boldsymbol{a} = [\underline{a}, \overline{a}]$ ordered by

$$\boldsymbol{a} \sqsubseteq \boldsymbol{b} \iff [\underline{a}, \overline{a}] \supseteq [\underline{b}, \overline{b}].$$

$(\mathbb{IR}, \sqsubseteq)$ is a domain referred to as the *partial real line*.
The *way–below* relation of $\mathbb{IR}$ is given by

$$\boldsymbol{a} \ll \boldsymbol{b} \quad \text{iff} \quad \underline{a} < \underline{b} \leq \overline{b} < \overline{a}.$$

The family

$$\left\{ \twoheaduparrow \boldsymbol{a} \mid \boldsymbol{a} \in \mathbb{IR}, \underline{a}, \overline{a} \in \mathbb{Q} \right\},$$

being $\twoheaduparrow \boldsymbol{a} = \{\boldsymbol{b} \in \mathbb{IR} \mid \boldsymbol{a} \ll \boldsymbol{b}\}$, forms a countable basis of the *Scott topology* $\mathcal{O}\mathbb{IR}$ on $(\mathbb{IR}, \sqsubseteq)$. We will always consider $\mathbb{IR}$ endowed with this topology.

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- *Projections:* Let $\pi_1, \pi_2 \colon \mathbb{IR} \to \mathbb{R}$ defined for each $\boldsymbol{a} \in \mathbb{IR}$ by

$$\pi_1(\boldsymbol{a}) = \underline{a} \quad \text{and} \quad \pi_2(\boldsymbol{a}) = \overline{a}.$$

  Note that π_1 is lower semicontinuous and π_2 is upper semicontinuous.

- Let $e \colon \mathbb{R} \to \mathbb{IR}$ defined by $a \mapsto e(a) = [a, a]$. This is an embedding. We shall identify

$$\mathbb{R} \simeq e(\mathbb{R}) \subseteq \mathbb{IR} \quad \text{and}$$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "e\circ f"'] & \mathbb{R} \arrow[d, "e"] \\
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We denote by $\mathrm{C}(X, \mathbb{IR})$ the collection of all continuous functions from X into $\mathbb{IR}$, the *continuous partial real functions*.

Lemma

$f\colon X \to \mathbb{IR}$ is continuous iff $\pi_1 \circ f$ is lower semicontinuous and $\pi_2 \circ f$ is upper semicontinuous.

Two partial orders on $\mathrm{C}(X, \mathbb{IR})$:

- $f \sqsubseteq g \iff f(\boldsymbol{a}) \sqsubseteq g(\boldsymbol{a})$ for all $\boldsymbol{a} \in \mathbb{IR}$;
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The frame of real numbers

We will denote by Frm the category with objects frames and morphisms frame homomorphisms.

Since Frm is an *algebraic category*, we can define frames by generators and relations, as in an algebraic fashion.

The frame of reals

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ the frame generated by all ordered pairs (p, q) of rationals, subject to the relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee \{(r, s) \mid p < r < s < q\}$,

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Definition

A *continuous real function* on a frame L is a frame homomorphism $h \colon \mathfrak{L}(\mathbb{R}) \to L$

We shall denote $\mathrm{C}(L) = \mathrm{Frm}(\mathfrak{L}(\mathbb{R}), L)$.

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The frame of reals (Equivalent definition)

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(r3) $(r, —) = \bigvee_{s>r}(s, —)$, for every $r \in \mathbb{Q}$,

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(r3) $(r, -) = \bigvee_{s > r}(s, -)$, for every $r \in \mathbb{Q}$,

(r4) $(-, r) = \bigvee_{s < r}(-, s)$, for every $r \in \mathbb{Q}$,

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With $(p, q) = (p, -) \wedge (-, q)$ one goes back to the previous definition.

The frame of reals (Equivalent definition)

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The frame of partial reals

When investigating the existence of suprema of families of continuous real functions on a frame one immediately realizes that the problem lies on the defining relation (r2) (or (R2)).

Definition

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Proposition

$\Sigma\mathfrak{L}(\mathbb{IR})$ is homeomorphic to the $\mathbb{IR}$ endowed with the Scott topology. The homeomorphism $\tau\colon \Sigma\mathfrak{L}(\mathbb{IR}) \to \mathbb{IR}$ is such that

$$\underline{\tau(\xi)} = \bigvee\{r \in \mathbb{Q} \mid \xi(r,-) = 1\} \quad \text{and}$$
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$$\Phi\colon \mathrm{Frm}(\mathfrak{L}(\mathbb{IR}), \mathcal{O}X) \xrightarrow{\sim} \mathrm{Top}(X, \mathbb{IR}).$$

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We shall denote by $\mathrm{IC}(L)$ the set $\mathrm{Frm}(\mathfrak{L}(\mathbb{IR}), L)$, partially ordered by

$$
\begin{array}{llll}
h_1 \leq h_2 & \text{iff} & h_1(r,—) \leq h_2(r,—) & \text{and} \\
 & & h_2(—,r) \leq h_1(—,r) & \text{for all } r \in \mathbb{Q} \text{ and} \\
h_1 \sqsubseteq h_2 & \text{iff} & h_1(r,—) \leq h_2(r,—) & \text{and} \\
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h \in \mathrm{C}(L) \longleftrightarrow \hat{h} \in \mathrm{IC}(L) & \text{such that} \\
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Proposition

$\mathrm{IC}(L)$ *is Dedekind order complete.*

- Let $\mathcal{H} = \{h_i\}_{i\in I} \subseteq \mathrm{IC}(L)$ be bounded from above.

$$\bigvee^{\mathrm{IC}(L)} \mathcal{H}(r,\text{—}) = \bigvee_{i\in I} h_i(r,\text{—}) \,, \quad \bigvee^{\mathrm{IC}(L)} \mathcal{H}(\text{—},s) = \bigvee_{q<s} \bigwedge_{i\in I} h_i(\text{—},q).$$

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Now, since $\mathrm{IC}(L)$ is Dedekind order complete it follows that it contains the Dedekind order completion of all its subsets, in particular $\mathrm{C}(L)$.

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Lemma

Let L be a completely regular frame and let $h \in \mathrm{IC}(L)$ be such that

(1) $\{f \in \mathrm{C}(L) \mid f \leq h\} \neq \varnothing$ *and*

(2) $h(p, -)^* \leq h(-, q)$ *whenever* $p < q$.

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Definitions

$$\mathrm{C}(L)^{\vee} = \{h \in \mathrm{IC}(L) \mid \exists f, g \in \mathrm{C}(L) : f \leq h \leq g \text{ and } h(p, \text{—})^* \leq h(\text{—}, q) \text{ if } p < q\},$$

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$$\mathrm{C}(L)^{\vee\!\!\!\wedge} = \mathrm{C}(L)^{\vee} \cap \mathrm{C}(L)^{\wedge}.$$

Corollary

Let L be a completely regular frame and let $h \in \mathrm{C}(L)^{\vee\!\!\!\wedge}$. Then

$$h = \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}(L) \mid f \leq h\} = \bigwedge^{\mathrm{IC}(L)} \{g \in \mathrm{C}(L) \mid h \leq g\}.$$

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$$\mathrm{C}(L)^{\mathbb{X}} = \mathrm{C}(L)^{\vee} \cap \mathrm{C}(L)^{\wedge}.$$

Corollary

Let L be a completely regular frame and let $h \in \mathrm{C}(L)^{\mathbb{X}}$. Then

$$h = \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}(L) \mid f \leq h\} = \bigwedge^{\mathrm{IC}(L)} \{g \in \mathrm{C}(L) \mid h \leq g\}.$$

Each arrow represents a strict inclusion:

$$\begin{tikzcd}
 & & \mathrm{C}(L)^{\vee} \arrow[dr] & \\
\mathrm{C}(L) \arrow[r] & \mathrm{C}(L)^{\vee\!\!\!\wedge} \arrow[ur] \arrow[dr] & & \mathrm{IC}(L) \\
 & & \mathrm{C}(L)^{\wedge} \arrow[ur] &
\end{tikzcd}$$

Proposition

Let L be a frame and $h \in \mathrm{IC}(L)$. Then $h \in \mathrm{C}^{\vee\!\!\!\wedge}(L)$ iff

(a) *there exist $f, g \in \mathrm{C}(L)$ such that $f \leq h \leq g$ and*

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Theorem

$\mathrm{C}(L)^{\between}$ is Dedekind order complete.

Corollary

Let L be a frame. Then the Dedekind order completion $\mathrm{C}(L)^{\#}$ of $\mathrm{C}(L)$ coincides with $\mathrm{C}(L)^{\between}$, i.e. the set of continuous partial functions, $h \in \mathrm{IC}(L)$ such that:

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Theorem

$\mathrm{C}(L)^{\rotatebox{90}{$\bowtie$}}$ is Dedekind order complete.

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The bounded case follows similarly.

Definition

$$h \text{ is bounded} \quad \text{iff} \quad \exists r \in \mathbb{Q} \text{ such that } h(-r, r) = 1.$$

$\mathrm{IC}^*(L)$ denotes the set of *bounded continuous partial real functions*.

$$\mathrm{C}^*(L)^{\curlywedge\!\!\curlyvee} = \mathrm{C}(L)^{\curlywedge\!\!\curlyvee} \cap \mathrm{IC}^*(L).$$

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For any completely regular frame L, $\mathrm{C}^*(L)^{\curlywedge\!\!\curlyvee}$ is the Dedekind order completion of $\mathrm{C}^*(L)$.

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$$h \text{ is bounded} \quad \text{iff} \quad \exists r \in \mathbb{Q} \text{ such that } h(-r, r) = 1.$$

$\mathrm{IC}^*(L)$ denotes the set of *bounded continuous partial real functions*.

$$\mathrm{C}^*(L)^{\text{\tiny X}} = \mathrm{C}(L)^{\text{\tiny X}} \cap \mathrm{IC}^*(L).$$

Proposition

For any completely regular frame L, $\mathrm{C}^*(L)^{\text{\tiny X}}$ is the Dedekind order completion of $\mathrm{C}^*(L)$.

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Corollary

For any completely regular frame L, the following are equivalent:

(1) *L is extremally disconnected.*
(2) $\mathrm{C}(L) = \mathrm{C}(L)^{\times}$.
(3) $\mathrm{C}(L)$ *is Dedekind order complete.*
(4) $\mathrm{C}(L)$ *is closed under non-void bounded suprema.*
(5) $\mathrm{C}^{*}(L)$ *is Dedekind order complete.*
(6) $\mathrm{C}^{*}(L)$ *is closed under non-void bounded suprema.*

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For any completely regular frame L, the following are equivalent:

(1) *L is extremally disconnected.*

(2) $\mathrm{C}(L) = \mathrm{C}(L)^{\mathbb{X}}$.

(3) $\mathrm{C}(L)$ *is Dedekind order complete.*

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(5) $\mathrm{C}^*(L)$ *is Dedekind order complete.*

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Corollary

For any completely regular frame L, the following are equivalent:

(1) *L is extremally disconnected.*

(2) $\mathrm{C}(L) = \mathrm{C}(L)^{\Join}$.

(3) $\mathrm{C}(L)$ *is Dedekind order complete.*

(4) $\mathrm{C}(L)$ *is closed under non-void bounded suprema.*

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Corollary

For any completely regular frame L, the following are equivalent:

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(2) $\mathrm{C}(L) = \mathrm{C}(L)^{\between}$.

(3) $\mathrm{C}(L)$ *is Dedekind order complete.*

(4) $\mathrm{C}(L)$ *is closed under non-void bounded suprema.*

(5) $\mathrm{C}^*(L)$ *is Dedekind order complete.*

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The classical case

$\mathrm{C}(X)^{\Join}$ denotes the set of functions $h \in \mathrm{C}(X, \mathbb{IR})$ such that

- there exist $f, g \in \mathrm{C}(X)$ such that $f \leq h \leq g$
- $h \sqsubseteq h' \implies h = h'$.

Corollary

For any completely regular topological space $(X, \mathcal{O}X)$, the following are equivalent:

(1) *X is extremally disconnected.*

(2) $\mathrm{C}(X) = \mathrm{C}(X)^{\Join}$.

(3) *$\mathrm{C}(X)$ is Dedekind order complete.*

(4) *$\mathrm{C}(X)$ is is closed under non-void bounded suprema.*

(5) $\mathrm{C}^*(X) = \mathrm{C}^*(X)^{\Join}$.

(6) *$\mathrm{C}^*(X)$ is Dedekind order complete;.*

(7) *$\mathrm{C}^*(X)$ is is closed under non-void bounded suprema.* $\square$

The classical case

$\mathrm{C}(X)^{\between}$ denotes the set of functions $h \in \mathrm{C}(X, \mathbb{IR})$ such that

- there exist $f, g \in \mathrm{C}(X)$ such that $f \leq h \leq g$
- $h \sqsubseteq h' \implies h = h'$.

Corollary

For any completely regular topological space $(X, \mathcal{O}X)$, the following are equivalent:

(1) *X is extremally disconnected.*

(2) $\mathrm{C}(X) = \mathrm{C}(X)^{\between}$.

(3) $\mathrm{C}(X)$ *is Dedekind order complete.*

(4) $\mathrm{C}(X)$ *is is closed under non-void bounded suprema.*

(5) $\mathrm{C}^*(X) = \mathrm{C}^*(X)^{\between}$.

(6) $\mathrm{C}^*(X)$ *is Dedekind order complete;.*

(7) $\mathrm{C}^*(X)$ *is is closed under non-void bounded suprema.* □

Eskerrik asko.

Thank you.