Workshop on Category Theory

In honour of George Janelidze, on the occasion of his 60th birthday

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Compact ordered spaces and semi-left-exactness

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We shall characterize a full subcategory of Nachbin's compact ordered spaces whose reflection into Priestley spaces is semi-left-exact (admissible, in the sense of categorical Galois theory). In order to do so we need the simplification given in [3] to the pullback preservation conditions in the definition of a semi-left-exact reflection (see [2]). Then we generalize the proofs in [4, 5.6, 5.7]; in particular, we work with an appropriate notion of connected component, and present a non-symmetrical generalization of entourage. Notice that the next step will be to try to extend the classical monotone-light factorization in compact Hausdorff spaces (with trivial orders) to this category of ordered spaces.

References

[1] Borceux, F., Janelidze, G., *Galois theories*, Cambridge University Press, 2001.

[2] Cassidy, C., Hébert, M., Kelly, G. M., *Reflective subcategories, localizations and factorization systems*, J. Austral. Math. Soc. 38**A** (1985) 287–329.

[3] João J. Xarez, *Generalising connected components*, J. Pure Appl. Algebra 216 (2012) 1823–1826.

[4] Nachbin, L., *Topology and Order*, Von Nostrand, Princeton, N. J., 1965.

<u>Ground structure</u> : $\quad \mathbb{S} \xleftarrow{U} \mathbb{C} \underset{H}{\overset{I}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbb{M}$, $\eta : 1_{\mathbb{C}} \to HI$ unit

$\mathcal{E} \subseteq \mathrm{mor}(\mathbb{S}) \qquad T \in \mathcal{T} \subseteq \mathrm{obj}(\mathbb{M})$

$\mathbb{C}$ has pullbacks ; H is a full inclusion ; U preserves pullbacks ;
$\mathcal{E}$ is pullback stable, closed under composition, and $f \in \mathcal{E} \wedge f'f \in \mathcal{E} \to f' \in \mathcal{E}$;
$\forall_{C \in \mathbb{C}} \quad U(\eta_C) : U(C) \to UHI(C)$ belongs to $\mathcal{E}$;
a morphism $g : N \to M$ in $\mathbb{M}$ is an isomorphism whenever $UH(g)$ is in $\mathcal{E}$ and there exists $f : A \to UH(N)$ in $\mathcal{E}$ such that, for every morphism $c : T \to M$ in $\mathbb{M}$ with T in $\mathcal{T}$, there exists a commutative diagram of the form

$$\begin{tikzcd}
A \times_{UH(M)} UH(T) \arrow[rr, two heads, "pr_2"] \arrow[dr, two heads, "l"] \arrow[dd, "pr_1"'] & & UH(T) \arrow[dd, "UH(c)"] \\
& UH(T) \arrow[ur] \arrow[d] & \\
A \arrow[r, two heads, "f"'] & UH(N) \arrow[r, two heads, "UH(g)"'] & UH(M)
\end{tikzcd}$$

with l in $\mathcal{E}$. $\qquad$ " $\twoheadrightarrow$ " means " $\in \mathcal{E}$ "

<u>theorem</u>. $\quad I \dashv H$ is semi-left-exact iff $HI(C_\mu) \cong T$, for every C_μ.

$$\begin{tikzcd}
C_\mu \arrow[r, "\pi_2^\mu"] \arrow[d, "\pi_1^\mu"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "\mu"] \\
C \arrow[r, "\eta_C"'] & HI(C)
\end{tikzcd}$$

(see "Generalising connected components", J. X., 2012)

If $I \dashv H$ is semi-left-exact and $p : E \to B$ is a morphism of Galois descent in $\mathbb{C}$, then :

$$\begin{tikzcd}
\mathrm{Spl}_B(E,p) \arrow[r, shift left, "p^*"] \arrow[r, phantom, "\top"] & \mathbb{R}_{I}/E \arrow[l, shift left, "\Sigma_p"] \arrow[r, shift left, "I^E"] \arrow[r, phantom, "\bot"] & \mathbb{M}/I(E) \arrow[l, shift left, "H^E"]
\end{tikzcd}$$

$I^E \circ p^*$ is monadic $\Rightarrow$ Galois theory

$$(\mathrm{Pre})\mathrm{Ord} \underset{\text{forgetful}}{\xleftarrow{\ U\ }} \underbrace{(\mathrm{Pre})\mathrm{Ord\,Top}}_{\text{complete category}} \underset{\overleftarrow{\ H\ }}{\overset{I}{\underset{\perp}{\longrightarrow}}} \mathrm{Ord\,T\,Dis} \quad \text{(H ~ full inclusion)}, \qquad X \xrightarrow[\eta_X]{} X/_{\mathcal{R}} \quad \text{quotient topology}$$

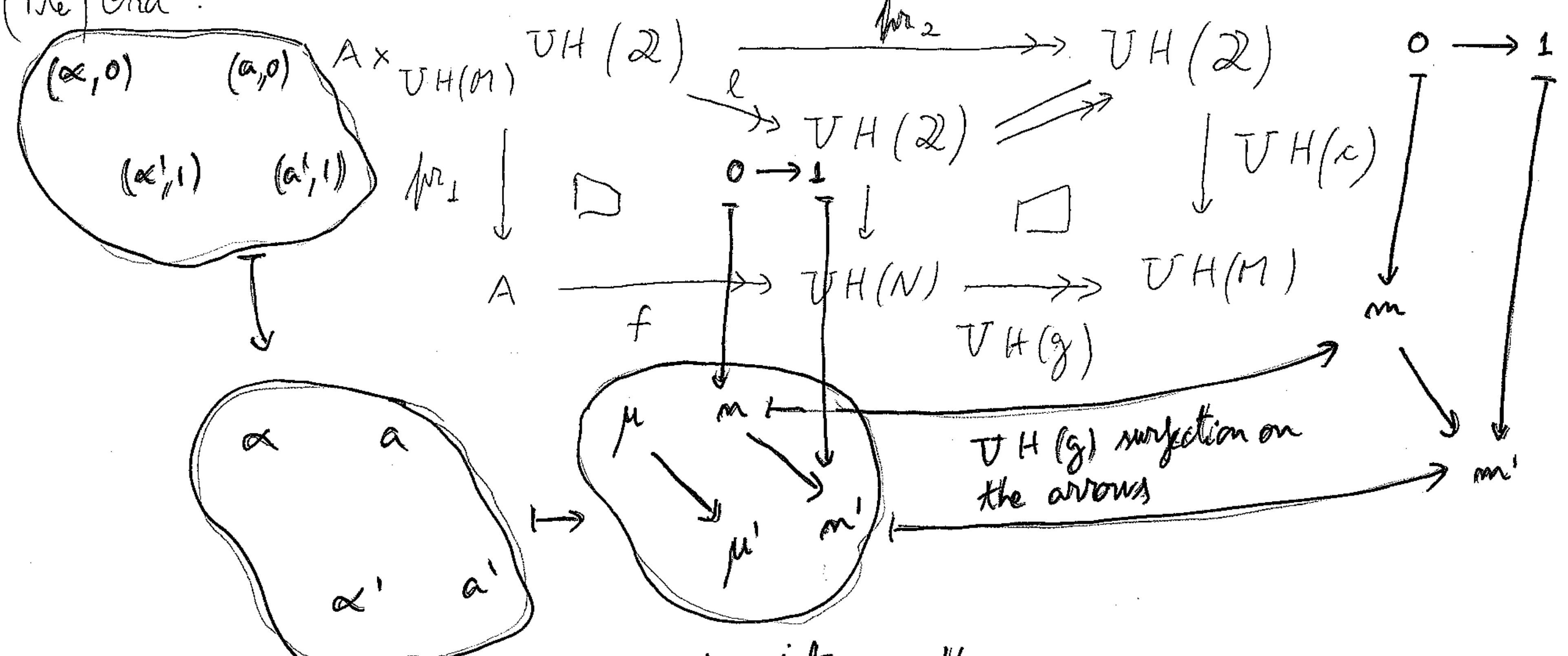
$\mathcal{E} = \{\text{surjections on the nodes}\}$ $\qquad \mathcal{I} = \{2\}$ $\qquad 2 = 0 \to 1$ with discrete topology

$\precsim$ preorder on the points of X : $x \,\mathcal{R}\, y \iff x \precsim y \wedge y \precsim x$;

$$x \precsim y \iff \forall_{\substack{U \subseteq X \\ \text{clopen upper set}}} \; x \in U \to y \in U \iff y \in \bigcap \{U \subseteq X \mid U \text{ clopen upper set}, x \in U\}$$

$$\iff \forall_{\substack{V \subseteq X \\ \text{clopen lower set}}} \; y \in V \to x \in V \iff x \in \bigcap \{V \subseteq X \mid V \text{ clopen lower set}, y \in V\}.$$

(Pre)Ord :



UH(g) injection on the arrows

$\therefore$ $UH(g)$ iso in $(\mathrm{Pre})\mathrm{Ord} \implies g$ iso in $\mathcal{M}$ <u>provided</u> <u>N compact</u>

Subreflection : $\underbrace{(\mathrm{Pre})\mathrm{Ord\,Comp}}_{\text{complete category}} \underset{\overleftarrow{\ H\ }}{\overset{I}{\underset{\perp}{\longrightarrow}}} \mathrm{Psp}$ (H ~ full inclusion)

$I \dashv H$ is semi-left-exact iff $HI(X_\mu) \cong 2$, for every X_μ

$$\begin{array}{ccc} X_\mu & \longrightarrow & 2 \\ \downarrow & & \downarrow \mu \\ X & \xrightarrow[\eta_X]{} & HI(X) \end{array} .$$

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Lemma Let $X \in$ (Pre) Ord Comp.

$$HI(\dot{X}_\mu) \cong 2 \text{ for every } \mu : 2 \to HI(X)$$

$$\text{iff} \Updownarrow$$

\begin{tikzcd}
X_\nu \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & HI(1) \arrow[d, "HI(\nu)"] \\
X \arrow[r, "\eta_X"] & HI(X)
\end{tikzcd}

$\eta_X : X \to HI(X)$ is a surjection on the (pre)order arrows,

provided $\boxed{HI(X_\nu) \cong 1 \text{ for every } \nu : 1 \to HI(X)}$ $(*)$.

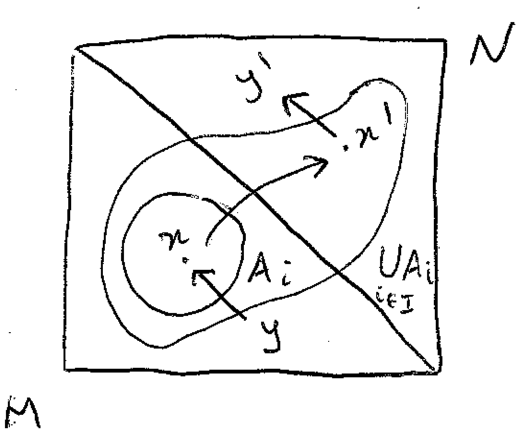
$$[x]_R \leq [y]_R \wedge \forall_{x \in [x]_R} \forall_{y \in [y]_R} x \nleq y \overset{(*)}{\Longleftrightarrow} I(X_\mu) \cong 2 \neq 2, \text{ with } \mu(2) = [x]_R \leq [y]_R$$

§ 5.7 in "Galois Theories", F. Borceux, G. Janelidze, 2001:

Definition 1. A (pre)ordered space is connected when it is nonempty and cannot be written as the union of two disjoint non-trivial open subsets, one of them a lower set and the other an upper set. A subset of X is connected when, provided with the induced order and topology, it is a connected space.

Lemma 2. $(A_i)_{i \in I}$ a family of connected subsets of X (pre)ordered space:

$$x \in \bigcap_{i \in I} A_i \neq \emptyset \Longrightarrow \bigcup_{i \in I} A_i \text{ connected}.$$



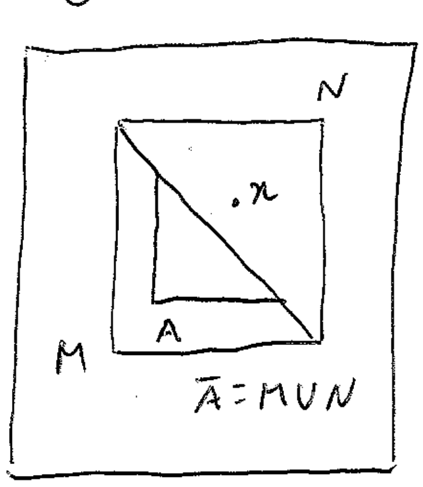
X not connected, $X = M \cup N$, $M \cap N = \emptyset$, M and N open non-void sets,

$$\forall_{x,y \in X} \left[(y \leq x \wedge x \in M) \to y \in M\right] \& \left[(x \leq y \wedge x \in N) \to y \in N\right]$$

Remarks: a connected component reduces to a point in every Priestley space; if the (pre)order is trivial this notion coincides with the classical one; finite discrete spaces with any (pre)order $\subseteq$ Psp.

__Definition 3.__ The connected component Γ_x of a point x in a (pre)ordered space X is the union of all connected subsets containing x, that is, the largest connected subset containing x. $\underset{\circlearrowleft}{\{x\}}$ is connected

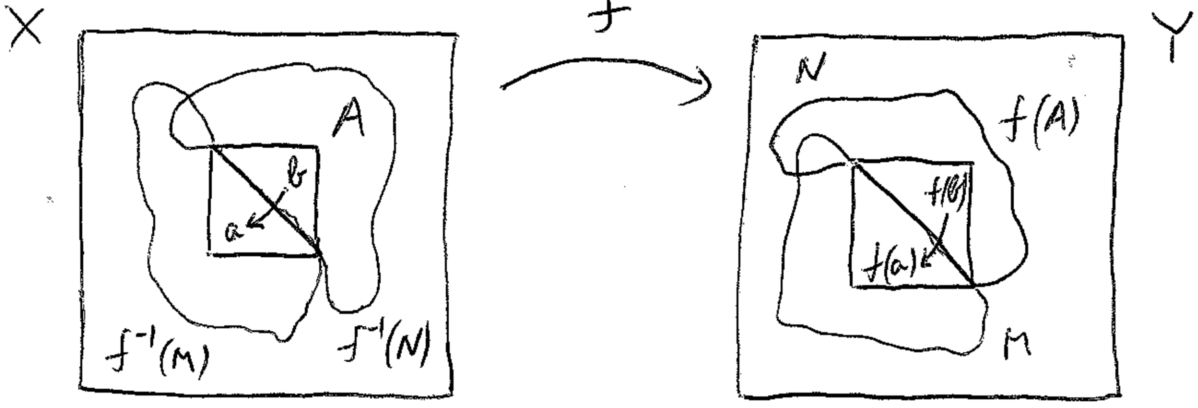
__Lemma 4.__ In a (pre)ordered space, the closure of a connected subset is again connected.



$\left.\begin{array}{l} x \in \bar{A} \\ x \in N \text{ open in } \bar{A} \end{array}\right\} \Rightarrow A \cap N \neq \emptyset$

__Corollary 5.__ In a (pre)ordered space, the connected component of a point is closed.

__Lemma 6.__ The image of a connected subset of a (pre)ordered space, under a continuous map preserving the order, is still connected.



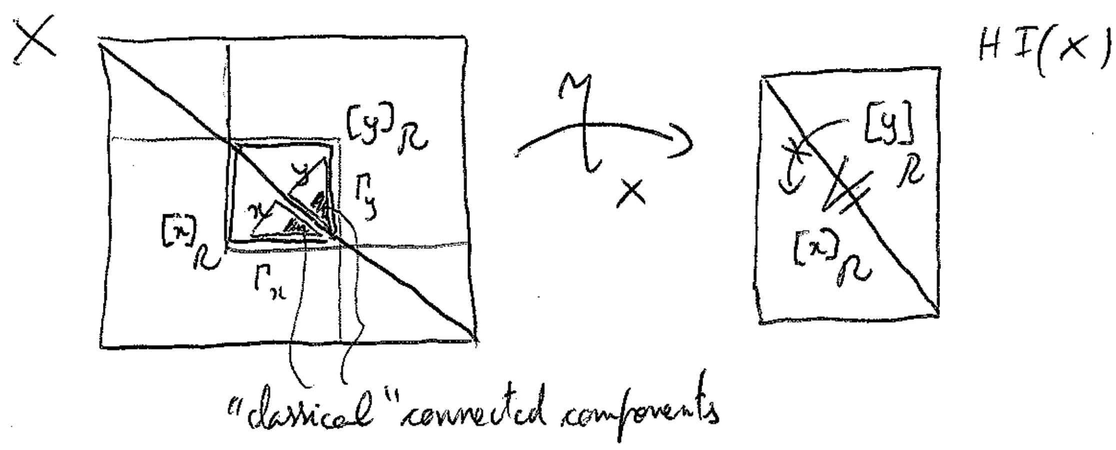
$$A = \underbrace{\left(A \cap f^{-1}(M)\right)}_{\substack{\text{clopen in } A \\ \text{lower set in } A}} \cup \underbrace{\left(A \cap f^{-1}(N)\right)}_{\substack{\text{clopen in } A \\ \text{upper set in } A}}$$

__Corollary 7.__ If $f : X \to Y$ is a morphism in (Pre)OrdTop and $x \in X$, then $f(\Gamma_x) \subseteq \Gamma_{f(x)}$.

$$X \xrightarrow{f} Y \in \mathrm{OrdTDis} \Rightarrow \forall_x \; f(\Gamma_x) = *$$

__Lemma 8.__ $x \in X \in$ (Pre)OrdTop:

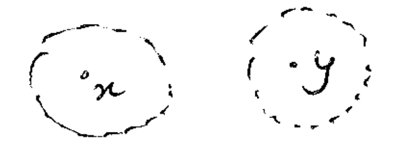
$$\Gamma_x \subseteq \bigcap \{ U \subseteq X \mid U \text{ clopen lower } \underline{\text{or}} \text{ upper set}, x \in U \} = \eta_X^{-1}\left(\{[x]_R\}\right)$$



__Definition 9.__ In a (pre)ordered topological space X, a neighbourhood of $\leq_X$ ($\subseteq X \times X$) is called an __entourage__ of X.

$\left(\begin{array}{c} N(\text{Pre})\text{Ord CompHaus} \\ \wr\!\wr \\ \text{Stably Compact (Dist)} \end{array} =\right)$ $\underbrace{N(\text{Pre})\text{ Ord Comp}}_{\text{complete category}} \underset{1+}{\overset{I}{\underset{\perp}{\rightleftarrows}}}$ ~~PsR~~

$x \not\leq y$:



Proposition 10. Let X be a Nachbin compact (pre)ordered space and $\mathcal{V}$ the set of its entourages. The following properties hold:

(i) $\forall_{V \in \mathcal{V}} \quad \leq_X \subseteq V$; (ii) $\forall_{V \in \mathcal{V}} \exists_{\substack{W \in \mathcal{V} \\ (\text{open})}} \quad W \circ W \subseteq V$.

Proof of (ii): reduction "ad absurdum". We fix $V \in \mathcal{V}$. $\forall_{W \in \mathcal{V}} \; W \circ W \not\subseteq V \Leftrightarrow W \circ W \cap V^c \neq \emptyset$.

$\mathcal{B} = \{W \circ W \cap V^c \mid W \in \mathcal{V}\}$ is a filter base on $X \times X$: $\emptyset \notin \mathcal{B}$; $V \circ V \cap V^c \in \mathcal{B}$;

$(W_1 \circ W_1 \cap V^c) \cap (W_2 \circ W_2 \cap V^c) \supseteq (W_1 \cap W_2) \circ (W_1 \cap W_2) \cap V^c \in \mathcal{B}$.

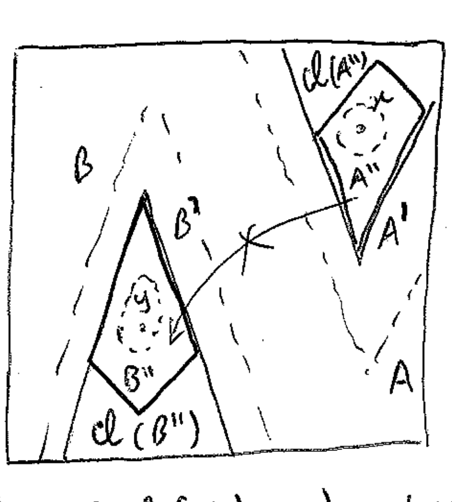
There is $(x,y) \in X \times X$ such that every neighbourhood meets every subset in the filter base (because $X \times X$ is compact Hausdorff) $\Rightarrow x \not\leq y$, because V is a neighbourhood of (x,y) if $x \leq y$

$\Rightarrow \exists_{\substack{A \text{ open} \\ \text{upper set}}} \exists_{\substack{B \text{ open} \\ \text{lower set}}} \; x \in A, y \in B, A \cap B = \emptyset$ (order separation) $\Rightarrow \exists_{\substack{A' \text{closed} \\ \text{upper set} \\ \text{neighb. of } x}} \exists_{\substack{B' \text{closed} \\ \text{lower set} \\ \text{neighb. of } y}} \; x \in A' \subseteq A, y \in B' \subseteq B, A \cap B = \emptyset$

$C := (A' \cup B')^c \qquad X = A \cup B \cup C \qquad \leq_X \subseteq W_0 = A \times A \cup B \times B \cup C \times C \cup C \times A \cup B \times C \cup B \times A$

$(u,v) \in A' \times B' \cap W_0 \circ W_0 \cap V^c \neq \emptyset \Rightarrow w \in A \cap B$ Contradiction $A' \ni u \xrightarrow{W_0} w \xrightarrow{W_0} v \in B'$ $\Longrightarrow \underset{A}{\cap} \; \underset{B}{\cap}$

$X = A \cup B \cup C$



- X is regular iff $\forall_{x \in X} \forall_{A \text{ open}} \; x \in A \to \exists_{A'' \text{open}} \; x \in A'' \subseteq cl(A'') \subseteq A$
- Every compact Hausdorff space is regular
- If C is a compact subset of a Nachbin's compact ordered space, then $\uparrow C$ and $\downarrow C$ are closed
- (Order separation) $x \not\leq y \to \exists_{\substack{A \text{ open} \\ \text{upper set}}} \exists_{\substack{B \text{ open} \\ \text{lower set}}} \; x \in A, y \in B, A \cap B = \emptyset$

$A' := \uparrow cl(A''), \; B' := \downarrow cl(B'')$

Lemma 11. Consider a subset $B \subseteq X$ of a (pre)ordered topological space X. For every open entourage V, we define

$$V^{\downarrow}(B) = \{x \in X \mid \exists_{b \in B} (x,b) \in V\} \text{ and } V^{\uparrow}(B) = \{x \in X \mid \exists_{b \in B} (b,x) \in V\}.$$

These subsets $V^{\downarrow}(B)$ and $V^{\uparrow}(B)$ are open in X.

$(-,b) : X \to X \times X$ continous, $x \mapsto (x,b)$ $\qquad V^{\downarrow}(B) = \bigcup_{b \in B} (-,b)^{-1}(V)$

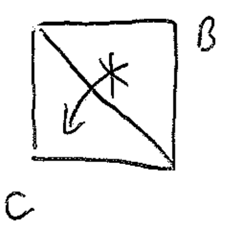
$(b,-) : X \to X \times X$ continous, $x \mapsto (b,x)$ $\qquad V^{\uparrow}(B) = \bigcup_{b \in B} (b,-)^{-1}(V)$

__Lemma 12__. Let B, C be __disjoint closed subspaces__ of a Nachbin's compact. (pre)ordered space X, and such that __B is an upper set__ and __C is a lower set__. There exists an __open__ entourage V such that

$$\left(V^{\uparrow}(B) \times V^{\downarrow}(C)\right) \cap V = \emptyset .$$

In particular, $V^{\uparrow}(B) \cap V^{\downarrow}(C) = \emptyset$.

Proof of Lemma 12. $B \times C$ closed, $B \times C \cap \leq_X = \emptyset$, $\leq_X$ closed



$\underset{\text{(normality)}}{\Rightarrow} B \times C \subseteq M$; $\leq_X \subsetneq N \in \mathcal{V}$; M, N open subsets of $X \times X$; $M \cap N = \emptyset$.

V open entourage s.t. $V \circ V \circ V \subseteq N$. Reduction "ad absurdum":

$(x, y) \in \left(V^{\uparrow}(B) \times V^{\downarrow}(C)\right) \cap V$ $\quad$ $x \in V^{\uparrow}(B) \Rightarrow \exists_{b \in B}\ (b, x) \in V$ $\quad$ $(x, y) \in V$

$y \in V^{\downarrow}(C) \Rightarrow \exists_{c \in C}\ (y, c) \in V$

$b \xrightarrow{V} x \xrightarrow{V} y \xrightarrow{V} c \Rightarrow (b, c) \in N$, $(b, c) \in B \times C \subseteq M$ Contradiction.

__Definition 13__. Let X be a Nachbin's compact (pre)ordered space.

(i) For an entourage V, the relation of V-nearness is the greatest equivalence relation on X contained in the transitive closure of V.

$[x]_V$ - class of x for the relation of V-nearness

(ii) the nearness relation on the space X is the intersection of all the V-nearness relations, for all entourages V.

$[x]_\sim$ - class of x for the nearness relation

__Lemma 14__. Let X be a (pre)ordered topological space. For every __open__ entourage V, the equivalence classes $[x]_V$ in X for the relation of V-nearness are clopens of X.

__Proof__. $x \in \underbrace{V^{\downarrow}(\{x\}) \cap V^{\uparrow}(\{x\})}_{\text{open}} = \{y \in X \mid (y, x), (x, y) \in V\} \subseteq \underset{\substack{\text{neighbourhood of each}\\ \text{of its points}}}{[x]_V}$

$\Rightarrow [x]_V$ is open in X $\Rightarrow [x]_V = \left(\bigcup_{y \not\sim_V x} [y]_V\right)^c$ clopen in X

$\left([x]_\sim = \bigcap \{[x]_V \mid V \text{ open entourage}\} \text{ closed in } X\right)$.

Lemma 15. In a (pre)ordered topological space X, the following two conditions are equivalent, for any $x, y \in X$:

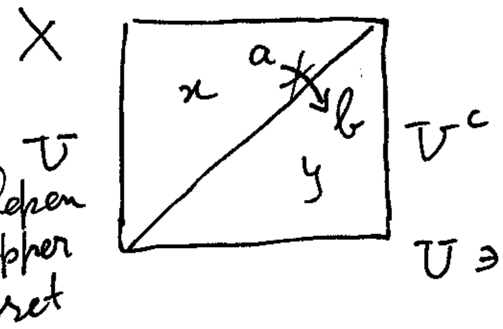
(a) $x \precsim y \quad \left(\Leftrightarrow y \in \bigcap \{ U \subseteq X \mid U \text{ clopen upper set}, x \in U \} \right)$;

(b) for every entourage V, there is a sequence

$$x = z_1, z_2, \ldots, z_{m-1}, z_m = y$$

for some $m \in \mathbb{N}$, such that $(z_i, z_{i+1}) \in V$ for all $i \in \{1, \ldots, m-1\}$.

Proof $b \Rightarrow a$: b & $\neg a$ (reduction "ad absurdum")



$$V := \bigcup_{(a,b) \in \leq_X} U_a \times U_b, \quad U_a = \begin{cases} U & \text{if } a \in U \\ U^c & \text{otherwise.} \end{cases}$$

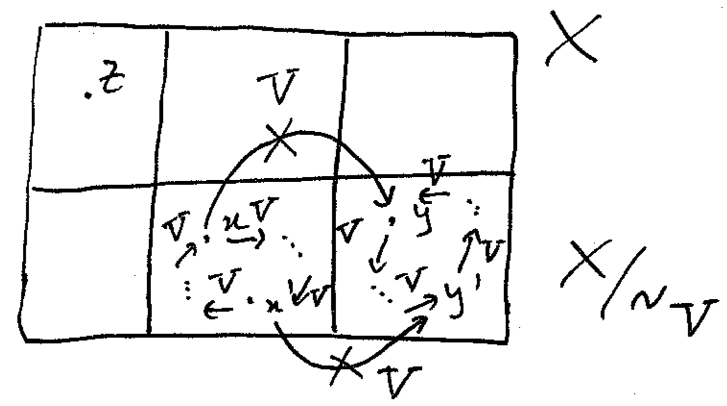
$$U \ni x = z_1 \xrightarrow{V} z_2 \xrightarrow{V} \cdots \xrightarrow{V} \underset{\overset{\cap}{U}}{z_i} \underset{\not\leq_X}{\xrightarrow{V}} \underset{\in U^c}{z_{i+1}} \xrightarrow{V} \cdots \xrightarrow{V} z_{m-1} \xrightarrow{V} z_m = y \in U^c$$

$(a,b) \in \leq_X \Rightarrow (a,b) \notin U \times U^c$, $(z_i, z_{i+1}) \in U \times U^c \Rightarrow (z_i, z_{i+1}) \notin V$ Contradiction

$(a \Rightarrow b \Leftrightarrow)$ $\neg b \Rightarrow \neg a$: $x \xrightarrow{V} \cdots \xrightarrow{V} y$ doesn't exist for some entourage V.

$S := \bigcup \{ [z]_V \mid \exists\, x \xrightarrow{V} \cdots \xrightarrow{V} z \} = \{ z \in X \mid \exists\, x \xrightarrow{V} \cdots \xrightarrow{V} z \}$ clopen upper set

$$\left. \begin{array}{l} x \in S \text{ clopen upper set} \\ y \notin S \end{array} \right\} \Rightarrow x \not\leq y$$



Corollary 16. Let x be a point of a (pre)ordered space X. Then,

$$[x]_\sim = [x]_{\mathcal{R}},$$

where $[x]_\sim = \bigcap \{ [x]_V \mid V \in \mathcal{V} \text{ and } V \text{ open} \}$, $[x]_{\mathcal{R}} = \bigcap \{ U \subseteq X \mid U \text{ clopen lower or upper set}, x \in U \}$.

Theorem 17. Let x be a point of a Nachbin's compact (pre)ordered space X, and Γ_x its connected component. Then,

$$\Gamma_x = [x]_{\mathcal{R}}.$$

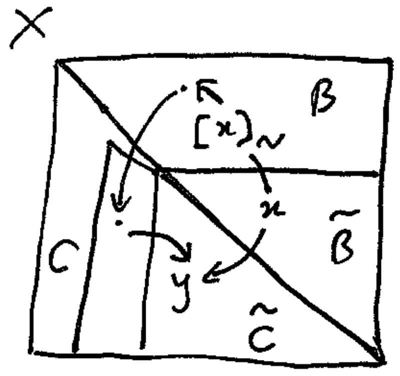
Proof of Theorem 17.

$$\underbrace{[x]_\sim \subseteq}_{\text{to be proved}} \Gamma_x \underbrace{\subseteq}_{\text{Lemma 8}} [x]_{\mathcal{R}} \underbrace{=}_{\text{Corollary 16}} [x]_\sim$$

$\cdot\ [x]_\sim \subseteq \Gamma_x$: reduction "ad absurdum" $[x]_\sim = \tilde{B} \cup \tilde{C}$, $\tilde{B} \cap \tilde{C} = \emptyset$,

$\tilde{B}$ upper set in $[x]_\sim$, $\tilde{C}$ lower set in $[x]_\sim$, $\tilde{B}$ and $\tilde{C}$ clopen in $[x]_\sim$,

$\tilde{B}$ and $\tilde{C}$ closed in X (because $[x]_\sim$ is closed in X) ;



$B := \uparrow\tilde{B}$, $C := \downarrow\tilde{C}$ closed in X ; $B \cap C = \emptyset$

$$\left(V^\uparrow(B) \times V^\downarrow(C)\right) \cap V = \emptyset \quad (\text{Lemma } 12)$$

$$\underbrace{W}_{\text{open in } X\times X} \subseteq W \circ W \subseteq V \Rightarrow \begin{cases} W^\uparrow(B) \subseteq V^\uparrow(B) \\ W^\downarrow(C) \subseteq V^\downarrow(C) \end{cases} \quad \big(\underbrace{W^\uparrow(B) \times W^\downarrow(C)}_{\text{open in } X}\big) \cap W = \emptyset$$

$$H := \left(W^\uparrow(B) \cup W^\downarrow(C)\right)^c \text{ closed in } X$$

$$x \sim y \Rightarrow \forall_{\underset{\text{open entourage}}{E \subseteq W}}\ x \sim_E y, \quad W^\uparrow(B) \supseteq \tilde{B} \ni x = z_1 \xrightarrow{E} z_2 \xrightarrow{E} \dots \xrightarrow{E} z_{m-1} \xrightarrow{E} z_m = y \in \tilde{C} \subseteq W^\downarrow(C)$$

At least <u>one z_i must be in H</u>, otherwise :

$$(*)\ \forall_i\ z_i \in H^c = W^\uparrow(B) \cup W^\downarrow(C) \subseteq V^\uparrow(B) \cup V^\downarrow(C)$$

$$z_i \in W^\uparrow(B) \Rightarrow \exists_{b_i \in B}\ (b_i, z_i) \in W \text{ and } (z_i, z_{i+1}) \in E \subseteq W$$

$$\Rightarrow \exists_{b_i \in B}\ (b_i, z_{i+1}) \in W \circ W \subseteq V$$

$$\Rightarrow z_{i+1} \in V^\uparrow(B) \overset{(*)}{\Rightarrow} z_{i+1} \in W^\uparrow(B) \quad \left(W^\uparrow(B) \cap W^\downarrow(C) = \emptyset\right)$$

<u>Filter basis</u> of closed subsets of $X = \{ [x]_E \cap H \mid \underset{\text{open}}{E} \subseteq W,\ E \in \mathcal{V} \}$

$\cdot\ [x]_E \cap H \neq \emptyset \ (\forall_i\ z_i \in [x]_E)$ $\quad\cdot\cdot\ [x]_{E \cap E'} \subseteq [x]_E \cap [x]_{E'}$

$x_0 \in \bigcap \{ [x]_E \cap H \mid E \text{ open}, E \subseteq W, E \in \mathcal{V} \} \neq \emptyset$ (since X is a compact Hausdorff space)

$\Rightarrow x_0 \in [x]_\sim \cap H$ Contradiction :

$$[x]_\sim = \tilde{B} \cup \tilde{C} \subseteq W^\uparrow(B) \cup W^\downarrow(C) = H^c$$

$\square$

$$\begin{tikzcd} N(\mathrm{Pre})\mathrm{OrdComp} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Psp} \arrow[l, shift left] \\ \mathbb{X} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Psp} \arrow[l, shift left] \arrow[u, equal] \\ \mathbb{C} \arrow[u, hook] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbb{M} \arrow[l, shift left] \arrow[u, hook] \end{tikzcd}$$

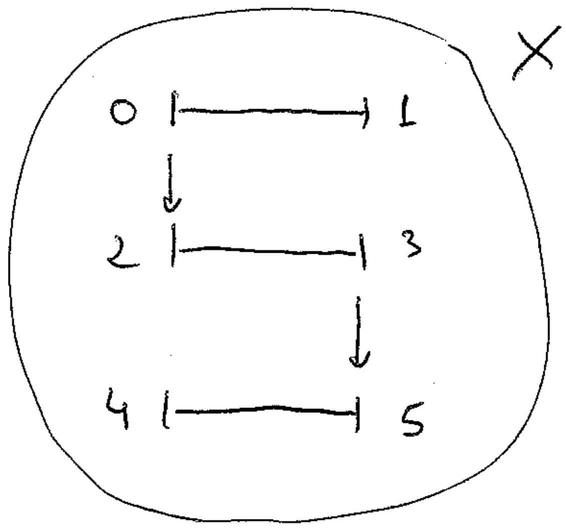
$\eta : 1_{N(\mathrm{Pre})\mathrm{OrdComp}} \longrightarrow HI$

$\mathrm{Obj}(\mathbb{X}) = \{ X \in N(\mathrm{Pre})\mathrm{OrdComp} \mid \eta_X$ surj. on the (pre)order arrows $\}$

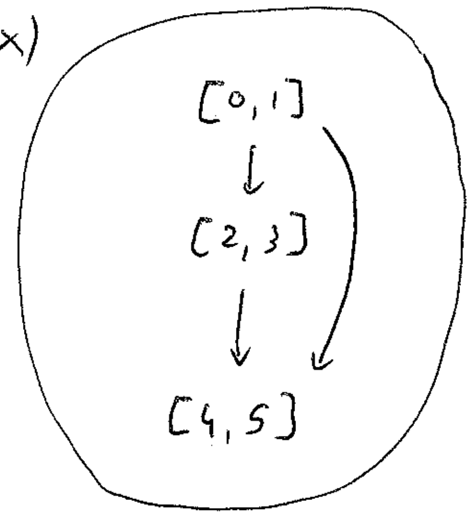
subreflection

$\mathbb{X} \neq N(\mathrm{Pre})\mathrm{OrdComp}$:

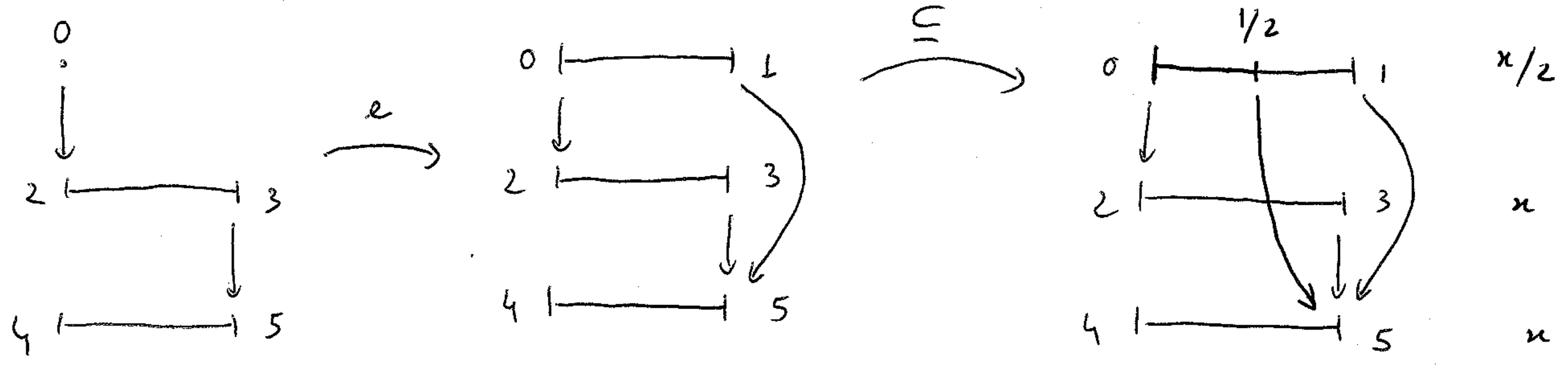
$\leq_X = \Delta \cup \{(0,2),(3,5)\}$



$X \xrightarrow{\eta_X} I(X)$



$\mathbb{X}$ doesn't have equalizers :



Examples ($\mathbb{C} \underset{\perp}{\rightleftarrows} \mathbb{M}$) :

CompHaus $\underset{\perp}{\rightleftarrows}$ Stone (A. Carboni, G. Janelidze, G.M. Kelly, R. Paré, 1997);

PPreord $\underset{\perp}{\rightleftarrows}$ Psp (M. Dias, 2004);

$\mathrm{obj}(\mathbb{C}) = \{ X \in \mathbb{X} \mid \leq_X$ equivalence relation closed in $X \times X \}$, $\mathbb{M} = \mathrm{Stone}$.