On some mysterious Mal’tsev conditions and the associated imaginary co-operations

*dedicated to George Janelidze*

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joint work with Diana Rodelo

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Some mysterious Mal’tsev conditions

Theorem [Hagemann & Mitschke, *On n-permutable congruences*, 1973]

For any equational class $\mathcal{V}$ and any $A \in \mathcal{V}$, the following are equivalent:

1. the congruence relations on $A$ are $n$-permutable;
2. every reflexive relation $R$ on $A$ satisfies $R^{\text{op}} \leq R^{n-1}$;
3. every reflexive relation $R$ on $A$ satisfies $R^n \leq R^{n-1}$.

The mystery

- Conditions 2 and 3 do not appear in [Carboni, Kelly & Pedicchio, *Some remarks on Maltsev and Goursat categories*, 1993]
- Nevertheless, all three conditions are purely categorical!
- We could, however, not find a categorical argument, and
- the proof Hagemann and Mitschke refer to was never published:
  [Hagemann, *Grundlagen der allgemeinen topologischen Algebra*, in preparation]

What’s going on?
Some mysterious Mal’tsev conditions

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What’s going on?
The associated imaginary co-operations

**Hagemann and Mitschke’s result is correct**

- 1 ⇔ 2 is treated in [Martins–Ferreira & VdL, 2010]
- 2 ⇔ 3 is also true for varieties

But what about general categories?

- the result holds in regular categories with finite sums
- proof technique mimics the varietal proof,
- based on Dominique Bourn and Zurab Janelidze’s
  approximate or imaginary co-operations
  [Bourn & Janelidze, Approximate Mal’tsev operations, 2008]

**Basic idea [Bourn & Janelidze, 2008]**

A Mal’tsev theory contains a Mal’tsev term \( p(x, y, z) \).
A regular Mal’tsev category has approximate Mal’tsev co-operations
\[
X \xleftarrow{\alpha_x} A(X) \xrightarrow{p_X} X + X + X
\]
which may be considered as *imaginary co-operations* \( p_X : X \longrightarrow 3X \).
The associated imaginary co-operations

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A Mal’tsev theory contains a Mal’tsev term $p(x, y, z)$.

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$$X \leftarrow \underbrace{\alpha_x}_{\text{opposite}} A(X) \xrightarrow{p_x} X + X + X$$

which may be considered as imaginary co-operations $p_X : X \rightsquigarrow 3X$. 
The associated imaginary co-operations

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“Whatever can be said about varieties can be proved categorically”
[Hans-E. Porst, yesterday]
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Overview

0 Introduction
1 Mal’tsev conditions
   ▷ The Mal’tsev case: 2-permutability
   ▷ The Goursat case: 3-permutability
   ▷ $n$-permutable categories
2 Imaginary co-operations
   ▷ Approximate Mal’tsev co-operations
   ▷ Approximate Goursat co-operations
   ▷ Main theorem: $n$-permutability
3 Conclusion
4 Further questions
The Mal’tsev case: 2-permutability

Theorem [Mal’tsev, 1954]

For any variety of algebras $\mathcal{V}$, the following are equivalent:

1. 2-permutability of congruences: $RS = SR$
2. existence of a ternary operation $p$ satisfying
   \[
   \begin{align*}
   p(x, y, y) &= x \\
   p(x, x, y) &= y
   \end{align*}
   \]

Such a $\mathcal{V}$ is called a **Mal’tsev variety**.

Theorem [Meisen, 1974; Faro, 1989; Carboni, Lambek & Pedicchio, 1990]

For any regular category $\mathcal{A}$, the following are equivalent:

1. 2-permutability of congruences: $RS = SR$
2. every reflexive relation $R$ is symmetric: $R^{\text{op}} \leq R$;
3. every reflexive relation $R$ is transitive: $R^2 \leq R$.

Such an $\mathcal{A}$ is called a (regular) **Mal’tsev category**.
The Mal’tsev case: 2-permutability

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For any variety of algebras $\mathcal{V}$, the following are equivalent:

1. 2-permutability of congruences: $RS = SR$
2. existence of a ternary operation $p$ satisfying

$$\begin{cases} p(x, y, y) = x \\ p(x, x, y) = y \end{cases}$$

Such a $\mathcal{V}$ is called a Mal’tsev variety.

Theorem [Meisen, 1974; Faro, 1989; Carboni, Lambek & Pedicchino, 1990]

For any regular category $\mathcal{A}$, the following are equivalent:

1. 2-permutability of congruences: $RS = SR$
2. every reflexive relation $R$ is symmetric: $R^{op} \leq R$; $R^{op} \leq R^{n-1}$
3. every reflexive relation $R$ is transitive: $R^2 \leq R$; $R^n \leq R^{n-1}$

Such an $\mathcal{A}$ is called a (regular) Mal’tsev category.
The Goursat case: 3-permutability

Theorem [Schmidt, 1969; Grötzer, Wille, 1970; Hagemann & Mitschke, 1973]

For any variety of algebras $\mathcal{V}$, the following are equivalent:

1. 3-permutability of congruences: $RSR = SRS$;
2. existence of quaternary operations $p$ and $q$ satisfying
   $$ p(x, y, y, z) = x, \quad p(x, x, y, y) = q(x, x, y, y), \quad q(x, y, y, z) = z; $$
3. existence of ternary operations $r$ and $s$ satisfying
   $$ r(x, y, y) = x, \quad r(x, x, y) = s(x, y, y), \quad s(x, x, y) = y; $$
4. every reflexive relation $R$ satisfies $R^\text{op} \leq R^2$;
5. every reflexive relation $R$ satisfies $R^3 \leq R^2$.

Such a $\mathcal{V}$ is called a 3-permutable or Goursat variety.

A regular category with 3-permutable congruences is called a (regular) Goursat category
[Carboni, Lambek & Pedicchio, 1990; Carboni, Kelly & Pedicchio, 1993].
The Goursat case: 3-permutability \( n = 3 \)

Theorem [Schmidt, 1969; Grötzer, Wille, 1970; Hagemann & Mitschke, 1973]

For any variety of algebras \( \mathcal{V} \), the following are equivalent:

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Theorem [Schmidt, 1969; Grötzer, Wille, 1970; Hagemann & Mitschke, 1973]

$\mathcal{V}$ is $n$-permutable when the following equivalent conditions hold:

1. $n$-permutability of congruences: $\overbrace{RSRS \cdots}^{n} = \overbrace{SRSR \cdots}^{n}$;

2. existence of $(n + 1)$-ary operations $v_0, \ldots, v_n$ satisfying

   \[
   \begin{align*}
   v_0(x_0, \ldots, x_n) &= x_0, & v_n(x_0, \ldots, x_n) &= x_n, \\
   v_{i-1}(x_0, x_0, x_2, x_2, \ldots) &= v_i(x_0, x_0, x_2, x_2, \ldots), & i & \text{ even}, \\
   v_{i-1}(x_0, x_1, x_1, x_3, x_3, \ldots) &= v_i(x_0, x_1, x_1, x_3, x_3, \ldots), & i & \text{ odd};
   \end{align*}
   \]

3. existence of ternary operations $w_1, \ldots, w_{n-1}$ satisfying

   \[
   \begin{align*}
   w_1(x, y, y) &= x, & w_{n-1}(x, x, y) &= y, \\
   w_i(x, x, y) &= w_{i+1}(x, y, y), & \text{ for } i & \in \{1, \ldots, n - 2\};
   \end{align*}
   \]

4. every reflexive relation $R$ satisfies $R^{\text{op}} \subseteq R^{n-1}$;

5. every reflexive relation $R$ satisfies $R^n \subseteq R^{n-1}$.

Notion of $n$-permutable category [Carboni, Kelly & Pedicchio, 1993].
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   ▶ The Goursat case: 3-permutability
   ▶ n-permutable categories

2 Imaginary co-operations
   ▶ Approximate Mal’tsev co-operations
   ▶ Approximate Goursat co-operations
   ▶ Main theorem: n-permutability

3 Conclusion

4 Further questions
Approximate Mal’tsev co-operations

Natural **approximate Mal’tsev co-operation** on \( \mathcal{A} \):

\[
\begin{align*}
X &\xrightarrow{\alpha_X} A(X) \\
\downarrow{\iota_1} &\quad &\uparrow{\iota_2} \\
2X &\xleftarrow{p_X} &2X \\
\downarrow{1_X + \nabla_X} &\quad &\downarrow{\nabla_X + 1_X} \\
3X
\end{align*}
\]

\[
\begin{align*}
\left\langle \frac{x}{x} \right\rangle \circ p_X &= y \circ \alpha_X \\
\left\langle \frac{x}{y} \right\rangle \circ p_X &= x \circ \alpha_X
\end{align*}
\]

**Universal** means \( A(X) \) limit of outer square

**Theorem** [Bourn & Janelidze, 2008]

Let \( \mathcal{A} \) be a regular category with binary coproducts. TFAE:

1. If \((\alpha, p)\) is universal, then \(\alpha\) is a regular epimorphism;
2. there exists an approximate Mal’tsev co-operation such that \(\alpha : A \Rightarrow 1_\mathcal{A}\) is a regular epimorphism;
3. \( \mathcal{A} \) is a Mal’tsev category.
Approximate Mal’tsev co-operations

Natural **approximate Mal’tsev co-operation** on $\mathcal{A}$:

![Diagram of approximate Mal’tsev co-operation](image)

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Approximate Mal’tsev co-operations

Natural approximate Mal’tsev co-operation on $\mathcal{A}$:

\[
\begin{align*}
X & \xrightarrow{\nu_1} A(X) \xrightarrow{\nu_2} X \\
2X & \xleftarrow{\rho_X} A(X) & 2X \\
1_x + \nabla_x & \xrightarrow{\rho_X} X & \nabla_x + 1_x \\
3X &
\end{align*}
\]

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Approximate Goursat co-operations

Natural approximate Goursat co-operations on $\mathcal{A}$:

Theorem

Let $\mathcal{A}$ be a regular category with binary coproducts. TFAE:

1. If $\alpha$ or $\beta$ is universal, then it is a regular epimorphism;
2. there exist approximate Goursat co-operations such that $\alpha$ and $\beta$ are regular epimorphisms;
3. $\mathcal{A}$ is a Goursat category;
4. every reflexive relation $R$ satisfies $R^{\text{op}} \subseteq R^2$. 

$\square$
Approximate Goursat co-operations

Natural approximate Goursat co-operations on $\mathcal{A}$:

\[
\begin{align*}
\text{quaternary} & : \\
& \\
\text{ternary} & : \\
\end{align*}
\]

Theorem

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4. every reflexive relation $R$ satisfies $R^{\text{op}} \leq R^2$. □

**What about condition 5?**

5. Every reflexive relation $R$ satisfies $R^3 \leq R^2$.

Follows from the characterisation of 4-permutability!
Approximate Goursat co-operations

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Main theorem: \( n \)-permutability

Natural **approximate ternary co-operations** on \( \mathcal{A} \), for \( n \geq 2 \):

\[\begin{array}{c}
\text{Theorem} \\
\text{Let } \mathcal{A} \text{ be a regular category with binary coproducts. TFAE:}
\end{array}\]

1. If \( \alpha \) or \( \beta \) is universal, then it is a regular epimorphism;
2. there exist approximate co-operations with \( \alpha \) and \( \beta \) regular epi;
3. \( \mathcal{A} \) is an \( n \)-permutable category.
Main theorem: $n$-permutability

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**Theorem**

A regular category with binary coproducts is $n$-permutable if and only if every reflexive relation $R$ satisfies $R^{\text{op}} \leq R^{n-1}$.

**Lemma**

If every reflexive relation $R$ in $\mathcal{A}$ satisfies $R^n \leq R^{n-1}$ then $\mathcal{A}$ is $(2n - 2)$-permutable.

**Theorem**

A regular category $\mathcal{A}$ with binary coproducts is $n$-permutable if and only if every reflexive relation $R$ satisfies $R^n \leq R^{n-1}$.

**Proof of $\Leftarrow$ in the Goursat case, $n = 3$.**

$R^3 \leq R^2$ implies that $\mathcal{A}$ is $2 \cdot 3 - 2 = 4$-permutable, so $R^{\text{op}} \leq R^{4-1} = R^3 \leq R^2 = R^{3-1}$, which gives $3$-permutability.
Main theorem: $n$-permutability

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Main theorem: $n$-permutability

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A regular category with binary coproducts is $n$-permutable if and only if every reflexive relation $R$ satisfies $R^{op} \leq R^{n-1}$. □

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Conclusion

- Hagemann and Mitschke’s theorem has a categorical counterpart:

**Theorem [Rodelo & VdL, 2012]**

For any regular category with binary sums $\mathcal{A}$ and any $A \in \mathcal{A}$, TFAE:

1. the equivalence relations on $A$ are $n$-permutable;
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Further questions

- Do we really need binary sums?
  - Counterexamples seem hard to construct:
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    - just “taking all finite algebras” or so will not work
  - Embedding theorem for $n$-permutable categories?
- Direct and simple “purely categorical” proof?
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- How general is this technique?
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The image contains a diagram with various nodes and arrows between them. The nodes are labeled with expressions related to $X$, $B(X)$, $2X$, $3X$, $1_x + \nabla_x$, $\nabla_x + 1_x$, $\nu_1$, $\nu_2$, $\beta_x$, $r_x$, and $s_x$. The arrows indicate the relationships between these expressions, with some dashed arrows suggesting a specific direction or relationship not fully specified in the diagram. The diagram appears to represent a mathematical or algebraic relationship, possibly in the context of some formal system or theory.