Generalised higher Hopf formulae for homology

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Outline

1. A description of the fundamental group in the semi-abelian context
2. A wider context
3. Generalized Higher Hopf formulae

General idea

Given a "good" adjunction

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend right, "H"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, bend right, "I"'] \end{tikzcd},$$

one can associate with any object B of $\mathcal{A}$ an invariant :

1. With a NORMAL EXTENSION $p\colon E \to B$ of B, one associates an object of $\mathcal{B}$: $\mathrm{Gal}(E, p, 0)$.
2. If p has a kind of UNIVERSAL PROPERTY, $\mathrm{Gal}(E, p, 0)$ is an invariant of B : $\pi_1(B)$ the abstract fundamental group of B.

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General idea

For the adjunction

$$\begin{tikzcd} \mathsf{Ab} \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathsf{Grp} \arrow[l, bend right, "\mathrm{ab}"'] \end{tikzcd}$$

$$\pi_1(B) \cong \frac{K \cap [P, P]}{[K, P]} \cong H_2(B, \mathbb{Z})$$

(for $K \to P \to B$ is a projective presentation of B) .

Galois structure [G. Janelidze]

Definition

A Galois structure is given by :

1. $\mathcal{B}$ a full replete reflective subcategory of $\mathcal{A}$

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend right, hook, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, bend right, "\mathrm{I}"'] \end{tikzcd} ;$$

2. $\mathcal{E}$ a class of morphisms in $\mathcal{A}$ which contains the isomorphisms of $\mathcal{A}$ and has some stability properties :
    1. $\mathrm{I}(\mathcal{E}) \subseteq \mathcal{E}$;
    2. $\mathcal{A}$ has pullback along morphisms in $\mathcal{E}$;
    3. $\mathcal{E}$ is closed under composition and pullback stable.

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Extensions

For a given Galois structure.

Definition

$f : E \to B$ in $\mathcal{E}$ is a ($\mathcal{B}$-)trivial extension if

$$\begin{tikzcd} E \arrow[r, "\eta_E"] \arrow[d, "f"'] & I(E) \arrow[d, "I(f)"] \\ B \arrow[r, "\eta_B"'] & I(B) \end{tikzcd}$$

is a pullback.

Extensions

Definition

An extension $f\colon A \to B$ is a ($\mathcal{B}$-)normal extension if in

$$\begin{tikzcd} A \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

π_1 and π_2 are trivial.

Extensions

For

$$
\begin{tikzcd}
\mathsf{Ab} \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathsf{Grp} \arrow[l, bend right, "\mathrm{ab}"']
\end{tikzcd}
$$

A regular epimorphism $f\colon A \to B$ is Ab-trivial iff

$$
\begin{tikzcd}
{[A,A]} \arrow[r, tail] \arrow[d, dotted, "\cong"'] & A \arrow[d, "f"] \\
{[B,B]} \arrow[r, tail] & B.
\end{tikzcd}
$$

A regular epimorphism $f\colon A \to B$ is Ab-normal iff it is central, i.e. if

$$\mathrm{Ker}\, f \subseteq Z(A).$$

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Extensions

Definition

A normal extension $p : E \to B$ is weakly universal if it factors through every other normal extension with the same codomain :

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[d, dotted, "\exists u"'] & \arrow[d, phantom, "\circlearrowleft"] & B \\ E' \arrow[rru, bend right, "p' \text{ normal}"'] & & \end{tikzcd}$$

The abstract fundamental group [G. Janelidze, 1984]

For a Galois structure with $\mathcal{A}$ semi-abelian

$$\begin{tikzcd} \mathcal{B} \arrow[rr, bend right, "\subseteq"'] \arrow[rr, phantom, "\perp"] & & \mathcal{A} \arrow[ll, bend right, "\mathrm{I}"'] \end{tikzcd}$$

and $p : E \to B$ a normal extension.

Definition

The Galois groupoid of p is :

$$\begin{tikzcd} \mathrm{I}((E \times_B E) \times_E (E \times_B E)) \arrow[r, "\mathrm{I}(\tau)"] & \mathrm{I}(E \times_B E) \arrow[loop above, "\mathrm{I}(\sigma)"] \arrow[r, shift left=2, "\mathrm{I}(\pi_1)"] \arrow[r, shift right=2, "\mathrm{I}(\pi_2)"'] & \mathrm{I}(E). \arrow[l, dotted, "\mathrm{I}(\delta)" description] \end{tikzcd}$$

The abstract fundamental group

Definition

The Galois group of p is defined via the following pullback :

$$\begin{tikzcd}
\mathsf{Gal}(E, p, 0) \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{I}(E \times_B E) \arrow[d, "\langle \mathsf{I}(\pi_1), \mathsf{I}(\pi_2) \rangle"] \\
0 \arrow[r, tail] & \mathsf{I}(E) \times \mathsf{I}(E).
\end{tikzcd}$$

The abstract fundamental group

Definition

The abstract fundamental group of an object B of B of $\mathcal{A}$ is the Galois group of any weakly universal normal extension of B.

A composite adjunction

One works with an adjunction

$$\begin{tikzcd} \mathcal{F} \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend right, "F"'] \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, bend right, "G"'] \end{tikzcd} \tag{A}$$

where

1. $\mathcal{A}$ is semi-abelian ;
2. $\mathcal{B}$ is a Birkhoff subcategory of $\mathcal{A}$;
3. $\mathcal{F}$ is a regular epi-reflective subcategory of $\mathcal{B}$;
4. F is protoadditive [T. Everaert and M. Gran, 2010] :
   F preserves split short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, tail, "k"] & A \arrow[r, shift right, two heads, "f"'] & B \arrow[l, shift right, "s"'] \arrow[r] & 0; \end{tikzcd}$$

and with $\mathcal{E} = \mathsf{RegEpi}(\mathcal{A})$.

Induced adjuncion

Theorem

One has an induced adjunction :

$$\begin{tikzcd} \mathrm{NExt}_{\mathcal{F}}(\mathcal{A}) \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathrm{Ext}(\mathcal{A}). \arrow[l, bend right, "F_1 \circ G_1"'] \end{tikzcd}$$

The reflection is given by

$$\begin{tikzcd} 0 \arrow[r] & {[f]_{1,\mathcal{F}}} \arrow[r, tail] \arrow[d, tail, "\alpha"'] & A \arrow[r, two heads] \arrow[d, equal] & \frac{A}{[f]_{1,\mathcal{F}}} \arrow[r] \arrow[d, dotted, two heads, "F_1G_1(f)"] & 0 \\ 0 \arrow[r] & \mathrm{Ker}\,(f) \arrow[r, tail, "\ker f"'] & A \arrow[r, two heads, "f"'] & B \arrow[r] & 0. \end{tikzcd}$$

Fröhlich construction

Construction

The Fröhlich construction is :

$$\begin{tikzcd}
{[f]_{1,\mathcal{F}}} \arrow[r, tail, "\ker(\hat{\pi}_1)"] \arrow[d, tail, dotted, "\alpha"'] \arrow[drr, tail, bend left] & \operatorname{Ker}(\eta_{A \times_B A}) \arrow[r, two heads, shift left, "\hat{\pi}_1"] \arrow[r, two heads, shift right, "\hat{\pi}_2"'] \arrow[d, tail] & \operatorname{Ker}(\eta_A) \arrow[d, tail] & \\
\operatorname{Ker}(f) \arrow[r, tail, "\ker \pi_1"'] & A \times_B A \arrow[r, two heads, shift left, "\pi_1"] \arrow[r, two heads, shift right, "\pi_2"'] & A \arrow[r, two heads, "f"] & B
\end{tikzcd}$$

where η is the unit.

Fröhlich construction

For

$$\begin{tikzcd} \mathrm{Ab} \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathrm{Grp} \arrow[l, bend right, "\mathrm{ab}"'] \end{tikzcd}$$

and $f\colon A \to B$ a regular epimorphism, one has

$$[f]_{1,\mathrm{Ab}} = [\mathrm{Ker}\,(f), A]_{\mathrm{Ab}} = \langle\, kak^{-1}a^{-1} \mid k \in \mathrm{Ker}\, f, a \in A \,\rangle = [\mathrm{Ker}\, f, A].$$

For

$$\begin{tikzcd} \mathrm{CRng} \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathrm{Rng} \arrow[l, bend right, "G"'] \end{tikzcd}$$

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$$[f]_{1,\mathrm{CRng}} = [\mathrm{ker}(f), A]_{\mathrm{CRng}} = \langle\, ak - ka \mid k \in \mathrm{Ker}\, f, a \in A \,\rangle.$$

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Construction of weakly universal normal extensions

Lemma

If $\mathcal{A}$ has enough projective objects w.r.t. $\mathcal{E}$, then for all B in $\mathcal{A}$ one can construct a weakly universal normal extension of B.

Proof : If $f : P \to B$ is a projective presentation of B :

$$
\begin{tikzcd}
P \arrow[rr, "f", triangle] \arrow[d] \arrow[drr, dotted, "\alpha"'] & & B \\
\tilde{P} \arrow[urr, "F_1G_1(f)"] \arrow[dr, dotted, "\beta"'] & & \\
& E \arrow[uur, "p\ \mathcal{F}\text{-normal}"'] &
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\end{tikzcd}$$

The generalized Hopf formula

Theorem

For $f : P \to B$ a projective presentation of B

$$\pi_1(B) \cong \frac{\overline{([P,P]_{\mathcal{B}})}^{\mathcal{F}}_P \cap \mathrm{Ker}\,(f)}{\overline{([\mathrm{Ker}\, f, P]_{\mathcal{B}})}^{\mathcal{F}}_{\mathrm{Ker}\, f}}.$$

$\overline{\cdot}^{\mathcal{F}}$ is a homological closure operator [D. Bourn and M. Gran, 2006].

Groups with coefficients in torsion free abelian groups

For

$$\begin{tikzcd} \mathrm{Ab}_{\mathrm{t.f.}} \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathrm{Ab} \arrow[l, bend right, "F"'] \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathrm{Grp} \arrow[l, bend right, "\mathrm{ab}"'] \end{tikzcd},$$

and a projective presentation $K \to P \to B$ of a group B,

$$\pi_1(B) \cong \frac{\{p \in K \mid \exists n \in \mathbb{N}_0 : p^n \in [P, P]\}}{\{p \in K \mid \exists n \in \mathbb{N}_0 : p^n \in [K, P]\}}.$$

Rings with coefficients in reduced commutative rings

For

$$\begin{tikzcd} \mathsf{RedCRng} \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathsf{CRng} \arrow[l, bend right, "F"'] \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathsf{Rng}, \arrow[l, bend right, "G"'] \end{tikzcd}$$

and a projective presentation $K \to P \to B$ of a ring B,

$$\pi_1(B) \cong \frac{\sqrt{[P,P]_{\mathsf{CRng}}}_{(P)} \cap K}{\sqrt{[K,P]_{\mathsf{CRng}}}_{(K)}}.$$

A wider context

$\mathsf{Grp}(\mathsf{Top})$ is regular but not exact.

But in some way, it is "almost exact" :

$$\begin{tikzcd} R \arrow[rr, tail, two heads] \arrow[dr, tail, "(r_1,r_2)"'] & & X \times_Q X \arrow[dl, tail, "(\pi_1,\pi_2)"] \\ & X \times X & \end{tikzcd}$$

where $q = \mathrm{coeq}(r_1, r_2) \colon X \longrightarrow Q$.

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where $q = \mathrm{coeq}(r_1, r_2) \colon X \longrightarrow Q$.

A wider context

For $\mathcal{A}$ with $(\mathbb{E}, \mathbb{M})$ a proper stable factorization system, finite limits and coequalizers of effective equivalence relations :

Definition

$\mathcal{A}$ is $\mathbb{E}$-exact if for every internal equivalence relation R there exist an effective equivalence relation S such that $R \leqslant_{\mathbb{E}} S$, i.e.

$$\begin{tikzcd} R \arrow[rr, tail, "i_{RS}"] \arrow[dr, tail, "(r_1,r_2)"'] & & S \arrow[dl, tail, "(s_1,s_2)"] \\ & X \times X & \end{tikzcd}$$

where i_{RS} is in $\mathbb{E}$.

($\mathbb{E}$-exact $\Rightarrow$ efficiently regular [D. Bourn, 2007]

$\Rightarrow$ almost Barr exact [G. Janelidze and M. Sobral, 2011])

A wider context

For our purpose, a good context to work in is the one of $\mathbb{E}$-exact homological categories.

Examples

1. All semi-abelian categories ($\mathbb{E} = \mathrm{RegEpi}$);
2. All topological semi-abelian varieties ($\mathbb{E} = \mathrm{Epi}$);
3. All integral almost abelian categories ($\mathbb{E} = \mathrm{Epi}$)
   $\approx$ Raïkov semi-abelian categories.

Topological groups with coefficients in Hausdorff Abelian groups

For

$$\begin{tikzcd} \mathrm{Ab}(\mathrm{Haus}) \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathrm{Ab}(\mathrm{Top}) \arrow[l, bend right, "F"'] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\perp"] & \mathrm{Grp}(\mathrm{Top}) \arrow[l, bend right, "\mathrm{ab}"'] \end{tikzcd}$$

and for $K \to P \to B$ a projective presentation of B,

$$\pi_1(B) \cong \frac{\overline{[P,P]}^{\mathrm{top}} \cap K}{\overline{[K,P]}^{\mathrm{top}}}.$$

Generalized higher Hopf formulae

One has still in the wider context an adjunction like

$$\mathsf{NExt}_{\mathcal{F}}(\mathcal{A}) \underset{\subseteq}{\overset{F_1 \circ G_1}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{Ext}(\mathcal{A}).$$

and then a Galois structure with the class of double extensions : squares $(f_1, f_0)\colon a \longrightarrow b$

$$\begin{tikzcd}
A_1 \arrow[dr, dotted, "{\langle a,f_1 \rangle}"] \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "a"'] & & \\
& A_0 \times_{B_0} B_1 \arrow[r] \arrow[d] & B_1 \arrow[d, "b"] \\
& A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

with all morphisms in $\mathcal{E} = \mathbb{E}$.

A general notion of closure operator [W. Tholen, 2011]

For $\mathcal{M}$ a class of monomorphisms in a category $\mathcal{A}$:

(a) containing isomorphisms ;

(b) closed under composition with isomorphisms ;

(c) satisfying the left-cancellation properties :

$$n \circ m \in \mathcal{M}, n \in \mathcal{M} \Rightarrow m \in \mathcal{M}.$$

and viewed as a full replete subcategory of $\mathrm{Arr}\mathcal{A}$.

Definition

A closure operator of $\mathcal{M}$ in $\mathcal{A}$ is an endofunctor $\overline{\cdot} : \mathcal{M} \longrightarrow \mathcal{M}$ such that :

(1) $\mathrm{Cod} = \mathrm{Cod} \circ \overline{\cdot}$;

(2) $\forall K \in \mathcal{M} : K \leqslant \overline{K}$;

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Generalized higher Hopf formulae

If P :

$$\begin{tikzcd} P_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & P_0 \arrow[d] \\ P_1 \arrow[r] & B \end{tikzcd}$$

is a 2-projective presentation of B, then

$$\pi_2(B) \cong \frac{\overline{([P_2, P_2]_{\mathcal{B}})}^{\mathcal{F}}_{P_2} \cap \mathrm{Ker}\,(p_1) \cap \mathrm{Ker}\,(p_2)}{\overline{([P]_{2,\mathcal{B}})}^{\mathcal{F}}_{\mathrm{Ker}\,(p_1) \cap \mathrm{Ker}\,(p_2)}}$$

where $[-]_{2,\mathcal{B}}$ is a kind of higher commutator.

Last example

For the adjunction

$$\mathrm{Ab}(\mathrm{Haus}) \underset{U}{\overset{F}{\leftrightarrows}} \mathrm{Ab}(\mathrm{Top}) \underset{H}{\overset{\mathrm{ab}}{\leftrightarrows}} \mathrm{Grp}(\mathrm{Top})$$

one has

$$\pi_2(B) \cong \frac{\overline{[P_2, P_2]}^{\mathrm{top}} \cap \mathrm{Ker}\,(p_1) \cap \mathrm{Ker}\,(p_2)}{\overline{[\mathrm{Ker}\,(p_1), \mathrm{Ker}\,(p_2)].[\mathrm{Ker}\,(p_1) \cap \mathrm{Ker}\,(p_2), P_2]}^{\mathrm{top}}}.$$

Thank you for your attention !