

Generalised higher Hopf formulae for homology

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Outline

- 1 A description of the fundamental group in the semi-abelian context
- 2 A wider context
- 3 Generalized Higher Hopf formulae

General idea

Given a “good” adjunction

$$\mathcal{B} \begin{array}{c} \xleftarrow{I} \\ \perp \\ \xrightarrow{H} \end{array} \mathcal{A},$$

one can associate with any object B of \mathcal{A} an invariant :

- 1 With a **NORMAL EXTENSION** $p: E \rightarrow B$ of B , one associates an object of \mathcal{B} : $\text{Gal}(E, p, 0)$.
- 2 If p has a kind of **UNIVERSAL PROPERTY**, $\text{Gal}(E, p, 0)$ is an invariant of B : $\pi_1(B)$ the **abstract fundamental group** of B .

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General idea

For the adjunction

$$\text{Ab} \begin{array}{c} \xleftarrow{\text{ab}} \\ \perp \\ \xrightarrow{\quad} \\ \subseteq \end{array} \text{Grp}$$

$$\pi_1(B) \cong \frac{K \cap [P, P]}{[K, P]} \cong H_2(B, \mathbb{Z})$$

(for $K \rightarrow P \rightarrow B$ is a projective presentation of B).

Galois structure [G. Janelidze]

Definition

A **Galois structure** is given by :

- 1 \mathcal{B} a full replete reflective subcategory of \mathcal{A}

$$\mathcal{B} \begin{array}{c} \xleftarrow{I} \\ \perp \\ \xrightarrow{\subseteq} \end{array} \mathcal{A};$$

- 2 \mathcal{E} a class of morphisms in \mathcal{A} which contains the isomorphisms of \mathcal{A} and has some stability properties :
 - 1 $I(\mathcal{E}) \subseteq \mathcal{E}$;
 - 2 \mathcal{A} has pullback along morphisms in \mathcal{E} ;
 - 3 \mathcal{E} is closed under composition and pullback stable.

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Extensions

For a given Galois structure.

Definition

$f : E \rightarrow B$ in \mathcal{E} is a (\mathcal{B} -)trivial extension if

$$\begin{array}{ccc} E & \xrightarrow{\eta_E} & I(E) \\ f \downarrow & & \downarrow I(f) \\ B & \xrightarrow{\eta_B} & I(B) \end{array}$$

is a pullback.

Extensions

Definition

An extension $f: A \rightarrow B$ is a (\mathcal{B} -)normal extension if in

$$\begin{array}{ccccc} A \times_B A & \xrightarrow{\pi_2} & A & & \\ \pi_1 \downarrow & \lrcorner & \downarrow f & & \\ A & \xrightarrow{f} & B & & \end{array}$$

π_1 and π_2 are trivial.

Extensions

For

$$\text{Ab} \begin{array}{c} \xleftarrow{\text{ab}} \\ \perp \\ \xrightarrow{\subseteq} \end{array} \text{Grp}$$

A regular epimorphism $f: A \rightarrow B$ is Ab-trivial iff

$$\begin{array}{ccc} [A, A] \triangleright \longrightarrow & A & \\ \cong \downarrow & & \downarrow f \\ [B, B] \triangleright \longrightarrow & B & \end{array}$$

A regular epimorphism $f: A \rightarrow B$ is Ab-normal iff it is central, i.e. if

$$\text{Ker } f \subseteq Z(A).$$

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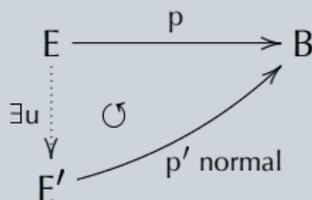
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Extensions

Definition

A **normal extension** $p: E \rightarrow B$ is **weakly universal** if it factors through every other normal extension with the same codomain :



The abstract fundamental group [G. Janelidze, 1984]

For a Galois structure with \mathcal{A} semi-abelian

$$\mathcal{B} \begin{array}{c} \xleftarrow{I} \\ \perp \\ \xrightarrow{I} \\ \subseteq \end{array} \mathcal{A}$$

and $p : E \rightarrow B$ a normal extension.

Definition

The **Galois groupoid** of p is :

$$I((E \times_B E) \times_E (E \times_B E)) \xrightarrow{I(\tau)} I(E \times_B E) \begin{array}{c} \overset{I(\sigma)}{\curvearrowright} \\ \xleftarrow{I(\pi_1)} \\ \xrightarrow{I(\pi_2)} \end{array} I(E).$$

$\xleftarrow{\dots I(\delta) \dots}$

The abstract fundamental group

Definition

The **Galois group** of p is defined via the following pullback :

$$\begin{array}{ccc} \text{Gal}(E, p, 0) & \xrightarrow{\quad} & I(E \times_B E) \\ \downarrow & \lrcorner & \downarrow \langle I(\pi_1), I(\pi_2) \rangle \\ 0 & \xrightarrow{\quad} & I(E) \times I(E). \end{array}$$

The abstract fundamental group

Definition

The **abstract fundamental group** of an object B of \mathcal{B} of \mathcal{A} is the Galois group of any weakly universal normal extension of B .

A composite adjunction

One works with an adjunction

$$\mathcal{F} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{\quad} \\ \subseteq \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{G} \\ \perp \\ \xrightarrow{\quad} \\ \subseteq \end{array} \mathcal{A} \quad (\text{A})$$

where

- 1 \mathcal{A} is **semi-abelian**;
- 2 \mathcal{B} is a **Birkhoff subcategory** of \mathcal{A} ;
- 3 \mathcal{F} is a regular epi-reflective subcategory of \mathcal{B} ;
- 4 F is **protoadditive** [T. Everaert and M. Gran, 2010] :
 F preserves split short exact sequences

$$0 \longrightarrow K \triangleright \xrightarrow{k} A \begin{array}{c} \xleftarrow{s} \\ \triangleright \\ \xrightarrow{f} \end{array} B \longrightarrow 0;$$

and with $\mathcal{E} = \text{RegEpi}(\mathcal{A})$.

Induced adjunction

Theorem

One has an induced adjunction :

$$\text{NExt}_{\mathcal{F}}(\mathcal{A}) \begin{array}{c} \xleftarrow{F_1 \circ G_1} \\ \perp \\ \xrightarrow{\subseteq} \end{array} \text{Ext}(\mathcal{A}).$$

The reflection is given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & [f]_{1, \mathcal{F}} & \twoheadrightarrow & A & \longrightarrow & \frac{A}{[f]_{1, \mathcal{F}}} \longrightarrow 0 \\ & & \downarrow \alpha & & \parallel & & \downarrow F_1 G_1(f) \\ 0 & \longrightarrow & \text{Ker}(f) & \xrightarrow{\text{ker } f} & A & \xrightarrow{f} & B \longrightarrow 0. \end{array}$$

Fröhlich construction

Construction

The Fröhlich construction is :

$$\begin{array}{ccccccc}
 [f]_{1, \mathcal{F}} & \xrightarrow{\ker(\hat{\pi}_1)} & \text{Ker}(\eta_{A \times_B A}) & \xrightarrow[\hat{\pi}_2]{\hat{\pi}_1} & \text{Ker}(\eta_A) & & \\
 \downarrow \alpha & & \downarrow & \searrow & \downarrow & & \\
 \text{Ker}(f) & \xrightarrow{\ker \pi_1} & A \times_B A & \xrightarrow[\pi_2]{\pi_1} & A & \xrightarrow{f} & B
 \end{array}$$

where η is the unit.

Fröhlich construction

For

$$\text{Ab} \begin{array}{c} \xleftarrow{\text{ab}} \\ \perp \\ \xrightarrow{\subseteq} \end{array} \text{Grp}$$

and $f: A \rightarrow B$ a regular epimorphism, one has

$$[f]_{1, \text{Ab}} = [\text{Ker}(f), A]_{\text{Ab}} = \langle kak^{-1}a^{-1} \mid k \in \text{Ker } f, a \in A \rangle = [\text{Ker } f, A].$$

For

$$\text{CRng} \begin{array}{c} \xleftarrow{G} \\ \perp \\ \xrightarrow{\subseteq} \end{array} \text{Rng}$$

and $f: A \rightarrow B$ a regular epimorphism, one has

$$[f]_{1, \text{CRng}} = [\text{ker}(f), A]_{\text{CRng}} = \langle ak - ka \mid k \in \text{Ker } f, a \in A \rangle.$$

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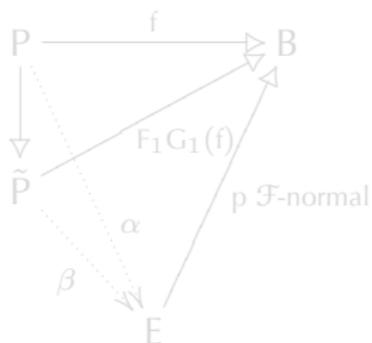
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Construction of weakly universal normal extensions

Lemma

If \mathcal{A} has enough projective objects w.r.t. \mathcal{E} , then for all B in \mathcal{A} one can construct a weakly universal normal extension of B .

Proof : If $f : P \rightarrow B$ is a projective presentation of B :

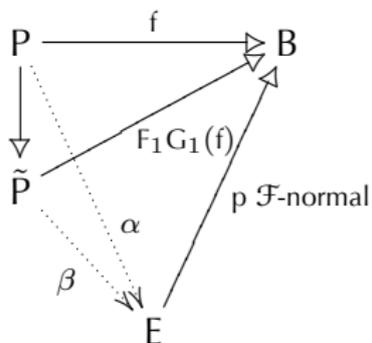


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The generalized Hopf formula

Theorem

For $f : P \rightarrow B$ a projective presentation of B

$$\pi_1(B) \cong \frac{\overline{([P, P]_{\mathcal{B}})_P}^{\mathcal{F}} \cap \text{Ker}(f)}{\overline{([\text{Ker } f, P]_{\mathcal{B}})_{\text{Ker } f}}^{\mathcal{F}}}.$$

$\overline{\cdot}^{\mathcal{F}}$ is a **homological closure operator** [D. Bourn and M. Gran, 2006].

Groups with coefficients in torsion free abelian groups

For

$$\text{Ab}_{\text{t.f.}} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{\quad} \\ \subseteq \end{array} \text{Ab} \begin{array}{c} \xleftarrow{\text{ab}} \\ \perp \\ \xrightarrow{\quad} \\ \subseteq \end{array} \text{Grp},$$

and a projective presentation $K \rightarrow P \rightarrow B$ of a group B ,

$$\pi_1(B) \cong \frac{\{p \in K \mid \exists n \in \mathbb{N}_0 : p^n \in [P, P]\}}{\{p \in K \mid \exists n \in \mathbb{N}_0 : p^n \in [K, P]\}}.$$

Rings with coefficients in reduced commutative rings

For

$$\text{RedCRng} \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{\subseteq} \end{array} \text{CRng} \begin{array}{c} \xleftarrow{G} \\ \perp \\ \xrightarrow{\subseteq} \end{array} \text{Rng},$$

and a projective presentation $K \rightarrow P \rightarrow B$ of a ring B ,

$$\pi_1(B) \cong \frac{\sqrt{[P, P]_{\text{CRng}(P)}} \cap K}{\sqrt{[K, P]_{\text{CRng}(K)}}}.$$

A wider context

$\text{Grp}(\text{Top})$ is regular but not exact.

But in some way, it is “almost exact” :

$$\begin{array}{ccc} R & \xrightarrow{\quad} & X \times_Q X \\ & \searrow (r_1, r_2) & \swarrow (\pi_1, \pi_2) \\ & X \times X & \end{array}$$

where $q = \text{coeq}(r_1, r_2): X \rightarrow Q$.

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A wider context

For \mathcal{A} with (\mathbb{E}, \mathbb{M}) a proper stable factorization system, finite limits and coequalizers of effective equivalence relations :

Definition

\mathcal{A} is \mathbb{E} -exact if for every internal equivalence relation R there exist an effective equivalence relation S such that $R \leq_{\mathbb{E}} S$, i.e.

$$\begin{array}{ccc} R & \xrightarrow{i_{RS}} & S \\ & \searrow (r_1, r_2) & \swarrow (s_1, s_2) \\ & X \times X & \end{array}$$

where i_{RS} is in \mathbb{E} .

- $(\mathbb{E}$ -exact \Rightarrow efficiently regular [D. Bourn, 2007])
- \Rightarrow almost Barr exact [G. Janelidze and M. Sobral, 2011])

A wider context

For our purpose, a **good context** to work in is the one of \mathbb{E} -exact homological categories.

Examples

- 1 All semi-abelian categories ($\mathbb{E} = \text{RegEpi}$);
- 2 All topological semi-abelian varieties ($\mathbb{E} = \text{Epi}$);
- 3 All integral almost abelian categories ($\mathbb{E} = \text{Epi}$)
 \approx Raïkov semi-abelian categories.

Topological groups with coefficients in Hausdorff Abelian groups

For

$$\text{Ab}(\text{Haus}) \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \text{Ab}(\text{Top}) \begin{array}{c} \xleftarrow{\text{ab}} \\ \perp \\ \xrightarrow{H} \end{array} \text{Grp}(\text{Top})$$

and for $K \rightarrow P \rightarrow B$ a projective presentation of B ,

$$\pi_1(B) \cong \frac{\overline{[P, P]}^{\text{top}} \cap K}{[K, P]^{\text{top}}}.$$

Generalized higher Hopf formulae

One has still in the wider context an adjunction like

$$\text{NExt}_{\mathcal{F}}(\mathcal{A}) \begin{array}{c} \xleftarrow{F_1 \circ G_1} \\ \perp \\ \xrightarrow{\subseteq} \end{array} \text{Ext}(\mathcal{A}).$$

and then a Galois structure with the class of **double extensions** : squares $(f_1, f_0) : a \longrightarrow b$

$$\begin{array}{ccccc} A_1 & & & & \\ & \searrow^{f_1} & & & \\ & & A_0 \times_{B_0} B_1 & \longrightarrow & B_1 \\ & \searrow^{\langle a, f_1 \rangle} & \downarrow & & \downarrow b \\ & & A_0 & \xrightarrow{f_0} & B_0 \\ & \searrow^a & & & \\ & & & & \end{array}$$

with all morphisms in $\mathcal{E} = \mathbb{E}$.

A general notion of closure operator [W. Tholen, 2011]

For \mathcal{M} a class of monomorphisms in a category \mathcal{A} :

- (a) containing isomorphisms ;
- (b) closed under composition with isomorphisms ;
- (c) satisfying the left-cancellation properties :

$$n \circ m \in \mathcal{M}, n \in \mathcal{M} \Rightarrow m \in \mathcal{M}.$$

and viewed as a full replete subcategory of $\text{Arr}\mathcal{A}$.

Definition

A **closure operator** of \mathcal{M} in \mathcal{A} is an endofunctor $\overline{\cdot} : \mathcal{M} \longrightarrow \mathcal{M}$ such that :

- (1) $\text{Cod} = \text{Cod} \circ \overline{\cdot}$;
- (2) $\forall K \in \mathcal{M} : K \leq \overline{K}$;
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Generalized higher Hopf formulae

If P :

$$\begin{array}{ccc} P_2 & \xrightarrow{p_2} & P_0 \\ p_1 \downarrow & & \downarrow \\ P_1 & \longrightarrow & B \end{array}$$

is a 2-projective presentation of B , then

$$\pi_2(B) \cong \frac{\overline{([P_2, P_2]_{\mathcal{B}})_{P_2}}^{\mathcal{F}} \cap \text{Ker}(p_1) \cap \text{Ker}(p_2)}{\overline{([P]_{2, \mathcal{B}})_{\text{Ker}(p_1) \cap \text{Ker}(p_2)}}^{\mathcal{F}}}$$

where $[-]_{2, \mathcal{B}}$ is a kind of higher commutator.

Last example

For the adjunction

$$\text{Ab}(\text{Haus}) \begin{array}{c} \xleftarrow{F} \\ \perp \\ \xrightarrow{U} \end{array} \text{Ab}(\text{Top}) \begin{array}{c} \xleftarrow{\text{ab}} \\ \perp \\ \xrightarrow{H} \end{array} \text{Grp}(\text{Top})$$

one has

$$\pi_2(\mathbb{B}) \cong \frac{\overline{[P_2, P_2]}^{\text{top}} \cap \text{Ker}(p_1) \cap \text{Ker}(p_2)}{[\text{Ker}(p_1), \text{Ker}(p_2)] \cdot [\text{Ker}(p_1) \cap \text{Ker}(p_2), P_2]^{\text{top}}}.$$

Thank you for your attention !