The manifestation of Hilbert's Nullstellensatz in Lawvere's Axiomatic Cohesion

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A science of cohesion

An explicit science of cohesion is needed to account for the varied background models for dynamical mathematical theories. Such a science needs to be sufficiently expressive to explain how these backgrounds are so different from other mathematical categories, and also different from one another and yet so united that can be mutually transformed.

F. W. Lawvere, Axiomatic Cohesion, TAC 2007

Cohesion and non-cohesion

The contrast of cohesion E with non-cohesion S can be expressed by geometric morphisms

$$p : E \to S$$

but that contrast can be made relative, so that S itself may be an 'arbitrary' topos.

Lawvere, TAC 2007

Axioms

Def. 1. A *topos of cohesion* (over $\mathcal{S}$) is:

$$\begin{tikzcd}
\mathcal{E} \arrow[d, bend right=60, "p_! \quad \dashv"'] \arrow[d, bend left=60, "p_* \quad \dashv"] \\
\mathcal{S} \arrow[u, "p^* \quad \dashv"']
\end{tikzcd}$$

such that ("The two downward functors express the opposition between 'points' and 'pieces'."):

Axioms

Def. 2. A *topos of cohesion* (over $\mathcal{S}$) is:

$$\begin{tikzcd}[row sep=large] \mathcal{E} \arrow[d, bend right=60, "p_! \;\dashv"'] \arrow[d, bend left=60, "p_* \;\dashv"] \\ \mathcal{S} \arrow[u, "p^* \;\dashv"'] \end{tikzcd}$$

such that (“The two downward functors express the opposition between ‘points’ and ‘pieces’.”):

1. $p^* : \mathcal{S} \to \mathcal{E}$ es full and faithful ($\Rightarrow \exists\, \theta : p_* \to p_!$).

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Def. 3. A *topos of cohesion* (over $\mathcal{S}$) is:

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such that ("The two downward functors express the opposition between 'points' and 'pieces'."):

1. $p^* : \mathcal{S} \to \mathcal{E}$ es full and faithful ($\Rightarrow \exists\, \theta : p_* \to p_!$).
2. $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products.

Axioms

Def. 4. A *topos of cohesion* (over $\mathcal{S}$) is:

$$\begin{tikzcd}[row sep=huge] \mathcal{E} \arrow[d, bend right=60, "p_!"'] \arrow[d, bend left=60, "p_* \ \dashv"] \\ \mathcal{S} \arrow[u, "\dashv\ p^*\ \dashv" description] \end{tikzcd}$$

such that (“The two downward functors express the opposition between ‘points’ and ‘pieces’.”):

1. $p^* : \mathcal{S} \to \mathcal{E}$ es full and faithful ($\Rightarrow \exists\, \theta : p_* \to p_!$).
2. $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products.
3. (Sufficient Cohesion) $p_!\Omega = 1$.

Axioms

Def. 5. A *topos of cohesion* (over $\mathcal{S}$) is:

$$\begin{tikzcd} \mathcal{E} \arrow[d, bend right=60, "p_!"'] \arrow[d, bend left=60, "p_*"] \\ \mathcal{S} \arrow[u, "p^*"] \end{tikzcd}$$

$$p_! \dashv p^* \dashv p_* \dashv$$

such that ("The two downward functors express the opposition between 'points' and 'pieces'."):

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2. $p_! : \mathcal{E} \to \mathcal{S}$ preserves finite products.
3. (Sufficient Cohesion) $p_!\Omega = 1$.
4. (Nullstellensatz) $\theta_X : p_*X \to p_!X$ is epi.

Example: Reflexive graphs

$$\begin{tikzcd} \widehat{\Delta_1} \arrow[d, bend right=60, "\pi_0"'] \arrow[d, bend left=60, "pts"] \\ \mathbf{Set} \arrow[u, "p^*"'] \end{tikzcd}$$

$$\pi_0 \dashv p^* \dashv pts$$

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1. $p^* : \mathbf{Set} \to \widehat{\Delta_1}$ is the full subcategory of ‘discrete graphs’ (the only edges are the identity loops).

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\end{tikzcd}$$

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2. π_0 preserves $\times$.

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1. $p^* : \mathbf{Set} \to \widehat{\Delta_1}$ is the full subcategory of ‘discrete graphs’ (the only edges are the identity loops).
2. π_0 preserves $\times$.
3. (Sufficient Cohesion) Ω is connected.

$$
\begin{tikzcd}
\bot \arrow[r, bend right] \arrow[loop above, dotted] & \top \arrow[l, bend right] \arrow[loop above, dotted] \arrow[loop below]
\end{tikzcd}
$$

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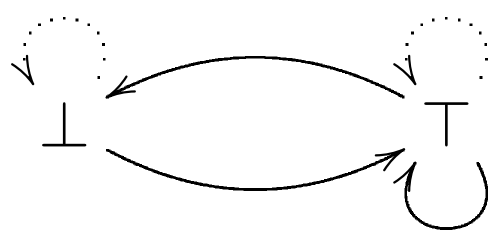

4. (Nullstellensatz) $\theta_X : \widehat{\Delta_1}(1, X) \to \pi_0 X$ is epi.

The Nullstellensatz over Set

Prop. 1 (Johnstone 2011). *Let small $\mathcal{C}$ have terminal object 1 so that the canonical $p^* : \mathbf{Set} \to \widehat{\mathcal{C}}$ below*

$$
\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \arrow[d, bend right=60, "p_!"'] \arrow[d, bend left=60, "p_*"] \arrow[r, phantom, "="] & \widehat{\mathcal{C}} \arrow[d, "\widehat{\mathcal{C}}(1,\text{-})"] \\
\mathbf{Set} \arrow[u, "p^*" description] \arrow[r, phantom, "="] & \widehat{1}
\end{tikzcd}
$$

$$p_! \dashv p^* \dashv p_* = \widehat{\mathcal{C}}(1,\text{-})$$

is full and faithful. Then

The Nullstellensatz over Set

Prop. 2 (Johnstone 2011). *Let small $\mathcal{C}$ have terminal object 1 so that the canonical $p^* : \mathbf{Set} \to \widehat{\mathcal{C}}$ below*

$$
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\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \arrow[d, bend right=60, "p_!"'] \arrow[d, bend left=60, "p_*"] & = & \widehat{\mathcal{C}} \arrow[d, "\widehat{\mathcal{C}}(1,\text{-})"] \\
\mathbf{Set} \arrow[u, "p^*" description] & = & \widehat{1}
\end{tikzcd}
\qquad p_! \dashv p^* \dashv p_* = \widehat{\mathcal{C}}(1,\text{-})
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is full and faithful. Then the Nullstellensatz holds if and only if every object C in $\mathcal{C}$ has a point $1 \to C$.

The Nullstellensatz over Set

Prop. 3 (Johnstone 2011). *Let small $\mathcal{C}$ have terminal object 1 so that the canonical $p^* : \mathbf{Set} \to \widehat{\mathcal{C}}$ below*

$$\begin{tikzcd}[row sep=large]
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\mathbf{Set} \arrow[u, "p^*"] & = & \widehat{1}
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is full and faithful. Then the Nullstellensatz holds if and only if every object C in $\mathcal{C}$ has a point $1 \to C$. Moreover, in this case, $p_!$ preserves finite products and $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is hyperconnected and local.

The Nullstellensatz over $\mathbf{Set}$

Prop. 4 (Johnstone 2011). *Let small $\mathcal{C}$ have terminal object 1 so that the canonical $p^* : \mathbf{Set} \to \widehat{\mathcal{C}}$ below*

$$\begin{tikzcd}[row sep=large]
\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \arrow[d, phantom, "p_! \dashv p^* \dashv p_*"] \arrow[d, bend right=60, "p_!"'] \arrow[d, bend left=60, "p_*"] & = & \widehat{\mathcal{C}} \arrow[d, "\widehat{\mathcal{C}}(1,\text{-})"] \\
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is full and faithful. Then the Nullstellensatz holds if and only if every object C in $\mathcal{C}$ has a point $1 \to C$. Moreover, in this case, $p_!$ preserves finite products and $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is hyperconnected and local.
(Addendum: Sufficient Cohesion holds iff some object of $\mathcal{C}$ has two distinct points.)

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Prop. 5 (Johnstone 2011). *Let small $\mathcal{C}$ have terminal object 1 so that the canonical $p^* : \mathbf{Set} \to \widehat{\mathcal{C}}$ below*

$$\begin{tikzcd}
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\mathbf{Set} \arrow[u, "p^*"] & = & \widehat{1}
\end{tikzcd}
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is full and faithful. Then the Nullstellensatz holds if and only if every object C in $\mathcal{C}$ has a point $1 \to C$. Moreover, in this case, $p_!$ preserves finite products and $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is hyperconnected and local.
(Addendum: Sufficient Cohesion holds iff some object of $\mathcal{C}$ has two distinct points.)

Ej. 5. Every object in $\Delta_1 = \begin{tikzcd} 1 \arrow[r, shift left=2] \arrow[r, shift right=2] & 2 \arrow[l] \end{tikzcd}$ has a point.

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Let k be an algebraically closed field. So that k-$\mathbf{Ext} = 1$.

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Def. 9. A k-algebra R is called *prime* if 0 and 1 are its only idempotents. Equivalently: $R = R_0 \times R_1$ implies $R_0 = 1$ o $R_1 = 1$.

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Let k-$\mathbf{Prime} \to k$-$\mathbf{Alg}_{fp}$ be the category of prime k-algebras.

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Let k be an algebraically closed field. So that $k\text{-}\mathbf{Ext} = 1$.

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Denote $(k\text{-}\mathbf{Alg}_{fp})^{\mathrm{op}}$ by $\mathbf{Aff}_k$, the cat. of ‘affine schemes’.

Def. 11. A k-algebra R is called *prime* if 0 and 1 are its only idempotents. Equivalently: $R = R_0 \times R_1$ implies $R_0 = 1$ o $R_1 = 1$.

Let $k\text{-}\mathbf{Prime} \to k\text{-}\mathbf{Alg}_{fp}$ be the category of prime k-algebras.

Prop. 11. *The induced subtopos* $[k\text{-}\mathbf{Prime}, \mathbf{Set}] \to [k\text{-}\mathbf{Alg}_{fp}, \mathbf{Set}]$ *satisfies*

$$\begin{tikzcd}
\mathbf{Aff}_k \arrow[dr, dotted, "\text{preserves } +"'] \arrow[drr, bend left, "\text{does not preserve } +"] & & \\
& \mathbf{Set}^{k\text{-}\mathbf{Prime}} \arrow[r] & \mathbf{Set}^{k\text{-}\mathbf{Alg}_{fp}}
\end{tikzcd}$$

Hilbert's Nullstellensatz (for closed k)

Teorema 1 (Hilbert's Nullstellensatz)**.** *If $A \in k$-$\mathbf{Prime}$ and $u \in A$ is not nilpotent then there is a $\chi : A \to k$ s.t. $\chi u \neq 0$.*

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Teorema 2 (Hilbert's Nullstellensatz)**.** *If $A \in k\text{-}\mathbf{Prime}$ and $u \in A$ is not nilpotent then there is a $\chi : A \to k$ s.t. $\chi u \neq 0$.*

Cor. 3. *So every object of $(k\text{-}\mathbf{Prime})^{\mathrm{op}}$ has a point.*

Hilbert's Nullstellensatz (for closed k)

Teorema 3 (Hilbert's Nullstellensatz). *If* $A \in k\text{-}\mathbf{Prime}$ *and* $u \in A$ *is not nilpotent then there is a* $\chi : A \to k$ *s.t.* $\chi u \neq 0$*.*

Cor. 5. *So every object of* $(k\text{-}\mathbf{Prime})^{\mathrm{op}}$ *has a point.*

Cor. 6. *The diagram*

$$
\begin{tikzcd}
\mathbf{Set}^{k\text{-}\mathbf{Prime}} \arrow[d, bend right=60, "p_! \;\dashv"'] \arrow[d, bend left=60, "\dashv\; p_*"] & & & \mathbf{Aff}_k \arrow[lll, "\text{Yoneda}"'] \\
\mathbf{Set} \arrow[u, "p^*" description] & = & \mathbf{Set}^{k\text{-}\mathbf{Ext}} & 1 \arrow[l, "\text{Yoneda}"'] \arrow[u]
\end{tikzcd}
$$

satisfies the Nullstellensatz.

Hilbert's Nullstellensatz (for closed k)

Teorema 4 (Hilbert's Nullstellensatz). *If $A \in k\text{-}\mathbf{Prime}$ and $u \in A$ is not nilpotent then there is a $\chi : A \to k$ s.t. $\chi u \neq 0$.*

Cor. 7. *So every object of $(k\text{-}\mathbf{Prime})^{\mathrm{op}}$ has a point.*

Cor. 8. *The diagram*

$$\begin{tikzcd}
\mathbf{Set}^{k\text{-}\mathbf{Prime}} \arrow[d, bend right, "p_!"'] \arrow[d, bend left, "p_*"] & & \mathbf{Aff}_k \arrow[ll, "\text{Yoneda}"'] \\
\mathbf{Set} \arrow[u, "p^*"] \arrow[r, phantom, "="] & \mathbf{Set}^{k\text{-}\mathbf{Ext}} & 1 \arrow[l, "\text{Yoneda}"'] \arrow[u]
\end{tikzcd}$$

$p_! \dashv p^* \dashv p_*$

satisfies the Nullstellensatz.

If k is not algebraically closed then $\mathbf{Set}^{k\text{-}\mathbf{Prime}} \to \mathbf{Set}$ exists but the Nullstellensatz does not hold. (Because it is not true that every object of $\mathbf{Aff}_k$ has a point.)

Cohesion over the Galois topos

For example, in a case E of algebraic geometry wherein spaces of all dimensions exist, S is usefully taken as a corresponding category of zero-dimensional spaces such as the Galois topos (of Barr-atomic sheaves on finite extensions of the ground field).

Lawvere, TAC 2007

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Consider the site $(\mathcal{D}, A)$ where A is the *atomic topology* (i.e. every map in $\mathcal{C}$ is a cover).

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Consider the site $(\mathcal{D}, A)$ where A is the *atomic topology* (i.e. every map in $\mathcal{C}$ is a cover).

$\mathbf{Shv}(\mathcal{D}, A)$ is an *atomic topos* and it classifies an algebraic closure of k (Barr-Diaconescu 1980).

The Galois topos of a field

Fix a (perfect) field k.

Let k-$\mathbf{Ext}$ be the category of finite field extensions of k.

Let $\mathcal{D} = (k\text{-}\mathbf{Ext})^{\mathrm{op}}$.

Consider the site $(\mathcal{D}, A)$ where A is the *atomic topology* (i.e. every map in $\mathcal{C}$ is a cover).

$\mathbf{Shv}(\mathcal{D}, A)$ is an *atomic topos* and it classifies an algebraic closure of k (Barr-Diaconescu 1980).

The category $\mathbf{Shv}(\mathcal{D}, A)$ is equivalent to the category of continuous actions of the profinite Galois group of the algebraic closure of k. (An action $G \times S \to S$ is *continuous* iff $(\forall s \in S)$ the stabilizer $\{g \in G \mid g \cdot s = s\}$ is open in G.)

Cohesion over a non-standard base

Let $\mathcal{C}$ be a small category.

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Any full subcategory $i : \mathcal{D} \to \mathcal{C}$ with left adjoint $\phi : \mathcal{C} \to \mathcal{D}$ induces a

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\begin{tikzcd}
\widehat{\mathcal{C}} \arrow[d, bend right=60, "p_!"'] \arrow[d, bend left=60, "p_*"] \\
\widehat{\mathcal{D}} \arrow[u, "p^*"]
\end{tikzcd}
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with full and faithful $p^* : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$.

Cohesion over a non-standard base

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Any full subcategory $i : \mathcal{D} \to \mathcal{C}$ with left adjoint $\phi : \mathcal{C} \to \mathcal{D}$ induces a

$$\begin{tikzcd}
\widehat{\mathcal{C}} \arrow[d, bend right=70, "p_!"'] \arrow[d, bend left=70, "p_*"] \\
\widehat{\mathcal{D}} \arrow[u, "p^*" description]
\end{tikzcd}$$

with full and faithful $p^* : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$.

What conditions allow us to extract a model of Axiomatic Cohesion out of this?

The primitive Nullstellensatz

Def. 12. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

The primitive Nullstellensatz

Def. 13. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

Prop. 13. *Assume that $\mathcal{D}$ can be equipped with the atomic topology.*

The primitive Nullstellensatz

Def. 14. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

Prop. 14. *Assume that $\mathcal{D}$ can be equipped with the atomic topology. Let $\phi : \mathcal{C} \to \mathcal{D}$ induce a locally connected geometric morphism.*

The primitive Nullstellensatz

Def. 15. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

Prop. 15. *Assume that $\mathcal{D}$ can be equipped with the atomic topology. Let $\phi : \mathcal{C} \to \mathcal{D}$ induce a locally connected geometric morphism. If ϕ has a full and faithful right adjoint $i : \mathcal{D} \to \mathcal{C}$ satisfying the primitive Nullstellensatz then*

The primitive Nullstellensatz

Def. 16. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

Prop. 16. *Assume that $\mathcal{D}$ can be equipped with the atomic topology. Let $\phi : \mathcal{C} \to \mathcal{D}$ induce a locally connected geometric morphism. If ϕ has a full and faithful right adjoint $i : \mathcal{D} \to \mathcal{C}$ satisfying the primitive Nullstellensatz then the topos $\mathcal{F}$ over $\mathbf{Shv}(\mathcal{D}, at)$ obtained by pulling back*

$$\begin{tikzcd}
\mathcal{F} \arrow[rr] \arrow[d, bend right, "q_!"'] \arrow[d, bend left, "q_*"] & & \widehat{\mathcal{C}} \arrow[d, bend right, "p_!"'] \arrow[d, bend left, "p_*"] \\
\mathbf{Shv}(\mathcal{D}, at) \arrow[u, "q^*"] \arrow[rr] & & \widehat{\mathcal{D}} \arrow[u, "p^*"]
\end{tikzcd}$$

satisfies the Nullstellensatz.

The primitive Nullstellensatz

Def. 17. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

Prop. 17. *Assume that $\mathcal{D}$ can be equipped with the atomic topology. Let $\phi : \mathcal{C} \to \mathcal{D}$ induce a locally connected geometric morphism. If ϕ has a full and faithful right adjoint $i : \mathcal{D} \to \mathcal{C}$ satisfying the primitive Nullstellensatz then the topos $\mathcal{F}$ over $\mathbf{Shv}(\mathcal{D}, at)$ obtained by pulling back*

$$
\begin{tikzcd}
\mathcal{F} \arrow[rr] \arrow[d, bend right=50, "q_!"'] \arrow[d, bend left=50, "q_*"] & & \widehat{\mathcal{C}} \arrow[d, bend right=50, "p_!"'] \arrow[d, bend left=50, "p_*"] \\
\mathbf{Shv}(\mathcal{D}, at) \arrow[rr] \arrow[u, "\dashv\; q^* \;\dashv" description] & & \widehat{\mathcal{D}} \arrow[u, "\dashv\; p^* \;\dashv" description]
\end{tikzcd}
$$

satisfies the Nullstellensatz. Moreover, if some object of $\mathcal{C}$ has two distinct points then Sufficient Cohesion also holds.

Examples of the primitive Nullstellensatz

Def. 18. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

Examples of the primitive Nullstellensatz

Def. 19. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

Ej. 7. If $\mathcal{C}$ has a terminal object the unique $\mathcal{C} \to 1$ has a full and faithful right adjoint $i : 1 \to \mathcal{C}$. This adjoint satisfies the primitive Nullstellensatz iff every object of $\mathcal{C}$ has point. (Hence, Johnstone's result.)

Examples of the primitive Nullstellensatz

Def. 20. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

Ej. 8. If $\mathcal{C}$ has a terminal object the unique $\mathcal{C} \to 1$ has a full and faithful right adjoint $i : 1 \to \mathcal{C}$. This adjoint satisfies the primitive Nullstellensatz iff every object of $\mathcal{C}$ has point. (Hence, Johnstone's result.)

Recall that k-$\mathbf{Prime}$ is the category of finitely presentable k-algebras without idempotents.

Examples of the primitive Nullstellensatz

Def. 21. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

Ej. 9. If $\mathcal{C}$ has a terminal object the unique $\mathcal{C} \to 1$ has a full and faithful right adjoint $i : 1 \to \mathcal{C}$. This adjoint satisfies the primitive Nullstellensatz iff every object of $\mathcal{C}$ has point. (Hence, Johnstone's result.)

Recall that k-$\mathbf{Prime}$ is the category of finitely presentable k-algebras without idempotents.

Teorema 8 (Nullstellensatz for arbitrary k)**.** *If $A \in k$-$\mathbf{Prime}$ and $u \in A$ is not nilpotent then there is a finite extension $k \to K$ and a map $\chi : A \to K$ s.t. $\chi u \neq 0$.*

Examples of the primitive Nullstellensatz

Def. 22. The functor $i : \mathcal{D} \to \mathcal{C}$ satisfies the *primitive Nullstellensatz* if for every C in $\mathcal{C}$ there is a map $iD \to C$ with D in $\mathcal{D}$.

Ej. 10. If $\mathcal{C}$ has a terminal object the unique $\mathcal{C} \to 1$ has a full and faithful right adjoint $i : 1 \to \mathcal{C}$. This adjoint satisfies the primitive Nullstellensatz iff every object of $\mathcal{C}$ has point. (Hence, Johnstone's result.)

Recall that k-$\mathbf{Prime}$ is the category of finitely presentable k-algebras without idempotents.

Teorema 9 (Nullstellensatz for arbitrary k). *If $A \in k$-$\mathbf{Prime}$ and $u \in A$ is not nilpotent then there is a finite extension $k \to K$ and a map $\chi : A \to K$ s.t. $\chi u \neq 0$.*

Cor. 13. *The full inclusion $(k\text{-}\mathbf{Ext})^{\mathrm{op}} \to (k\text{-}\mathbf{Prime})^{\mathrm{op}}$ satisfies the primitive Nullstellensatz.*

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We can take the pullback

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathbf{Set}^{k\text{-}\mathbf{Prime}} \\ {\scriptstyle q_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathbf{Shv}(k\text{-}\mathbf{Ext}^{\mathrm{op}}, at) & \longrightarrow & \mathbf{Set}^{k\text{-}\mathbf{Ext}} \end{array}$$

and conclude:

... , for perfect k:

Cor. 14. *The Nullstellensatz holds for*

$$
\begin{tikzcd}
\mathcal{F} \arrow[d, bend right=60, "q_!"'] \arrow[d, bend left=60, "q_*"] \\
\mathbf{Shv}(k\text{-}\mathbf{Ext}^{\mathrm{op}}, at) \arrow[u, "q^*"]
\end{tikzcd}
\qquad q_! \dashv q^* \dashv q_*
$$

... , for perfect k:

Cor. 15. *The Nullstellensatz holds for*

$$
\begin{tikzcd}
\mathcal{F} \arrow[d, bend right=60, "q_!"'] \arrow[d, bend left=60, "q_*"] \\
\mathbf{Shv}(k\text{-}\mathbf{Ext}^{\mathrm{op}}, at) \arrow[u, "q^*" description] \arrow[u, phantom, shift right=4ex, "\dashv"] \arrow[u, phantom, shift left=4ex, "\dashv"]
\end{tikzcd}
$$

Moreover, Sufficient Cohesion holds and

... , for perfect k:

Cor. 16. *The Nullstellensatz holds for*

$$\begin{tikzcd}
\mathcal{F} \arrow[d, bend right=60, "q_!"'] \arrow[d, bend left=60, "q_*"] \\
\mathbf{Shv}(k\text{-}\mathbf{Ext}^{\mathrm{op}}, at) \arrow[u, "\dashv\ q^*\ \dashv" description]
\end{tikzcd}$$

Moreover, Sufficient Cohesion holds and
the category $\mathbf{Aff}_k$ *of affine spaces embeds into* $\mathcal{F}$ *preserving* $+$ *and finite limits.*

... , for perfect k:

Cor. 17. *The Nullstellensatz holds for*

$$\begin{tikzcd} \mathcal{F} \arrow[d, bend right=60, "q_!"'] \arrow[d, bend left=60, "q_*"] \arrow[d, phantom, shift right=3ex, "\dashv"] \arrow[d, phantom, shift left=3ex, "\dashv"] \\ \mathbf{Shv}(k\text{-}\mathbf{Ext}^{\mathrm{op}}, at) \arrow[u, "q^*"] \end{tikzcd}$$

Moreover, Sufficient Cohesion holds and the category $\mathbf{Aff}_k$ of affine spaces embeds into $\mathcal{F}$ preserving $+$ and finite limits.

I.e.: a model of Axiomatic Cohesion for algebraic geometry over a perfect field k.

The End.

Thanks.

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