A push forward construction and the comprehensive factorization for internal crossed modules II

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Overview

- The push forward and the comprehensive factorization in $\mathbf{XMod}(\mathcal{C})$
- A glance at the non-pointed context
- Some applications to butterflies

Notice. For the purposes of this talk, *category* means *finitely complete* and *finitely cocomplete category*.

1. The push forward construction and the comprehensive factorization in $\mathbf{XMod}(\mathcal{C})$

The comprehensive factorization

- In 1973 Street and Walters studied the factorization system on **Cat** which gives rise to the comprehension scheme for the Lawvere doctrine
$$P\colon \mathbf{Cat}^{op} \to \mathbf{Cat}$$
$$X \mapsto \mathbf{Set}^X$$
This is called the *comprehensive factorization of a functor*.
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Definition: A functor $F\colon \mathbb{A} \to \mathbb{B}$ is called *final* when for any discrete fibration G and commutative square as below, there exists a unique diagonal making the triangles commute:

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\forall"] \arrow[d, "F"'] & \mathbb{C} \arrow[d, "G \in Df"] \\ \mathbb{B} \arrow[r, "\forall"'] \arrow[ur, dashed, "\exists !"] & \mathbb{D} \end{tikzcd}$$

Theorem [Bourn 1987] Let $\mathcal{C}$ be Barr-exact. Given a morphism $P\colon \mathbb{H} \to \mathbb{G}$ in **Gpd**$(\mathcal{C})$, there is a unique, up to isomorphism, factorization $P = D \circ \widetilde{P}$, with D a discrete fibration and $\widetilde{P}$ a final functor.

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- In 2003 G. Janelidze introduced the notion of internal crossed module in a semi-abelian category.

Theorem [Janelidze 2003] Let $\mathcal{C}$ be a semi-abelian category. There is an equivalence of categories

$$\mathbf{XMod}(\mathcal{C}) \simeq \mathbf{Cat}(\mathcal{C})$$

$$\begin{tikzcd} G_0 \flat G \arrow[d, "\xi"] & & G \rtimes G_0 \arrow[dd, shift right=3, "d"'] \arrow[dd, shift left=3, "c"] \\ G \arrow[d, "\partial"] & \mapsto & \\ G_0 & & G_0 \arrow[uu, "e"] \end{tikzcd}$$

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The comprehensive factorization

Discrete fibrations have a simple description, in the language of crossed modules: they are morphisms (g, g_0) with g an iso.

$$\begin{tikzcd}
\widetilde{G} \arrow[r, "\simeq", "g"'] \arrow[d, "\widetilde{\partial}"'] & G \arrow[d, "\partial"] \\
\widetilde{G}_0 \arrow[r, "g_0"'] & G_0
\end{tikzcd}$$

What about final functors?

The push forward construction can help in answering this question.

Notice: we call *final* the morphisms of crossed modules corresponding to final functors.

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Notation

Lemma/Notation In a semi-abelian category $\mathcal{C}$ we consider the following morphism of split extensions

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & D \arrow[r, shift left, "p'"] \arrow[d, "f'"'] & E \arrow[l, shift left, "s'"] \arrow[d, "f"] \\
X \arrow[r] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"]
\end{tikzcd}
$$

Then

- the square on the right is a split pullback
- if $\xi\colon B\flat X \to X$ is the action determined by p on its kernel, then $f^*(\xi)$:

$$E\flat X \xrightarrow{f\flat 1_X} B\flat X \xrightarrow{\xi} X$$

coincides with the action determined by p' on its kernel.

Push forward of crossed modules

In a semi-abelian category $\mathcal{C}$, we consider the following data:

$$\begin{tikzcd} H_0\flat H \arrow[r, "\xi"] & H \arrow[r, "\partial"] \arrow[d, "p"] & H_0 \\ H_0\flat G \arrow[r, "\alpha"'] & G & \end{tikzcd}$$

- a crossed module $\mathbb{H} = (\partial\colon H \to H_0, \xi)$
- an arrow $p\colon H \to G$
- an internal action $\alpha\colon H_0\flat G \to G$

The push forward construction gives a universal way to "extend" the crossed module ∂ along the map p.

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Push forward of crossed modules

Some conditions have to be fulfilled:

$$
\begin{tikzcd}
H_0 \flat H \arrow[r, "\xi"] \arrow[d, "1\flat p"'] & H \arrow[d, "p"] \\
H_0 \flat G \arrow[r, "\alpha"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
(H \rtimes_\xi H_0)\flat G \arrow[r, "\varphi\flat 1"] \arrow[d, "(p\rtimes 1)\flat 1"'] & H_0 \flat G \arrow[d, "\alpha"] \\
(G \rtimes_\alpha H_0)\flat G \arrow[r, "\chi_|"'] & G
\end{tikzcd}
$$

must commute, where

- $\varphi\colon H \rtimes_\xi H_0 \to H_0$ is the domain map of $\mathbb{H}$
- $\chi_| = \chi_G^{G\rtimes H_0}$ is the conjugation action

Push forward of crossed modules

Then there exist

$$\begin{tikzcd} H_0 \flat H \arrow[r, "\xi"] \arrow[d, "\widetilde{p}_0 \flat p"'] & H \arrow[r, "\partial"] \arrow[d, "p"] & H_0 \arrow[d, "\widetilde{p}_0"] \\ \widetilde{G}_0 \flat G \arrow[r, "\widetilde{\xi}"'] & G \arrow[r, "\widetilde{\partial}"'] & \widetilde{G}_0 \end{tikzcd}$$

- a $\widetilde{\mathbb{G}} = (\widetilde{\partial}\colon G \to \widetilde{G}_0, \widetilde{\xi})$, with $\widetilde{G}_0 = G \rtimes_{\alpha}^{H} H_0$
- a morphism of crossed modules

$$\widetilde{P}\colon (p, \widetilde{p}_0)\colon \mathbb{H} \to \widetilde{\mathbb{G}}$$

such that $\widetilde{p}_0^*(\widetilde{\xi}) = \alpha$.

Push forward of crossed modules

Moreover this construction is universal with respect to all morphisms of crossed modules

- underlying the same p
- inducing by pullback the same action α

$$
\begin{tikzcd}
H \arrow[r, "\partial"] \arrow[d, "p"'] \arrow[dr, phantom, "p.f."] & H_0 \arrow[d, "\widetilde{p}_0"] \arrow[ddr, bend left, "p_0"] & \\
G \arrow[r, "\widetilde{\partial}"'] \arrow[dr, equal] & \widetilde{G}_0 \arrow[dr, dashed, "g_0"] & \\
& G \arrow[r, "\partial'"'] & G_0
\end{tikzcd}
$$

A remarkable property of the push forward

Proposition In a push forward diagram

$$\begin{tikzcd}
\mathrm{Ker}(\partial) \arrow[r] \arrow[d, "p_k"'] & H \arrow[r, "\partial"] \arrow[d, "p"'] & H_0 \arrow[r, two heads] \arrow[d, "\widetilde{p}_0"] & \mathrm{Coker}(\partial) \arrow[d, "\simeq"] \\
\mathrm{Ker}(\widetilde{\partial}) \arrow[r] & G \arrow[r, "\widetilde{\partial}"'] & \widetilde{G}_0 \arrow[r, two heads] & \mathrm{Coker}(\widetilde{\partial})
\end{tikzcd}$$

- the induced map on cokernels is an iso
- the restriction map p_K on kernels is a regular epi, with $\mathrm{Ker}(p_K) = \mathrm{Ker}(p) \cap \mathrm{Ker}(\partial)$.

A remarkable property of the push forward

Consequences: $p(\mathrm{Ker}(\partial)) = \mathrm{Ker}(\widetilde{\partial})$,
and the following...

$$\begin{tikzcd}
\mathrm{Ker}(\partial) \arrow[r] \arrow[d, two heads, "p_k"'] & H \arrow[r, "\partial"] \arrow[d, "p"'] & H_0 \arrow[r, two heads] \arrow[d, "\widetilde{p}_0"] & \mathrm{Coker}(\partial) \arrow[d, "\simeq"] \\
\mathrm{Ker}(\widetilde{\partial}) \arrow[r] \arrow[d, two heads] & G \arrow[r, "\widetilde{\partial}"'] \arrow[d, two heads, "\kappa"'] & \widetilde{G}_0 \arrow[r, two heads] \arrow[d, equal] & \mathrm{Coker}(\widetilde{\partial}) \arrow[d, equal] \\
0 \arrow[r] & \widetilde{\partial}(G) \arrow[r, tail, "\iota"'] & \widetilde{G}_0 \arrow[r, two heads] & \mathrm{Coker}(\iota)
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Consequences: $p(\mathrm{Ker}(\partial)) = \mathrm{Ker}(\widetilde{\partial})$,
and the following is a push forward:

$$
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\mathrm{Ker}(\partial) \arrow[r] \arrow[d] & H \arrow[r, "\partial"] \arrow[d, "\kappa p"'] \arrow[dr, phantom, "\text{p.f.}"] & H_0 \arrow[r, two heads] \arrow[d, "\widetilde{p}_0"] & \mathrm{Coker}(\partial) \arrow[d, "\simeq"] \\
0 \arrow[r] & \widetilde{\partial}(G) \arrow[r, tail, "\iota"'] & \widetilde{G}_0 \arrow[r, two heads] & \mathrm{Coker}(\iota)
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$$

The push forward inside a morphism

A special situation is determined by the push forward construction developed *inside* a given morphism of crossed modules.

Consider the morphism $P = (p, p_0) \colon \mathbb{H} \to \mathbb{G}$:

$$\begin{tikzcd} H \arrow[r, "p"] \arrow[d, "(\partial{,}\xi)"'] & G \arrow[d, "(\partial'{,}\xi')"] \\ H_0 \arrow[r, "p_0"'] & G_0 \end{tikzcd}$$

The action $p_0^*(\xi')$ *always* satisfies the conditions for the construction of the push forward of ∂ along p.

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The push forward inside a morphism

The universal property thus yields a factorization of P:

$$\begin{tikzcd} H \arrow[r, "p"'] \arrow[rr, bend left, "p"] \arrow[d, "\partial"'] & G \arrow[r, equal] \arrow[d, "\widetilde{\partial}"'] & G \arrow[d, "\partial'"] \\ H_0 \arrow[r, "\widetilde{p}_0"] \arrow[rr, bend right, "p_0"'] & \widetilde{G}_0 \arrow[r, "g_0"] & G_0 \end{tikzcd}$$

Theorem Let $\mathcal{C}$ be semi-abelian, and $P = (p, p_0) \colon \mathbb{H} \to \mathbb{G}$ a morphism in $\mathbf{XMod}(\mathcal{C})$. T.F.A.E.

- P is the push forward of ∂ along p, w.r.t. the action $p^*(\xi)$
- P is final

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The push forward and final functors

Thus the push forward gives a way to actually perform the final/discrete fibration construction for morphisms of crossed modules

$$
\begin{tikzcd}
H \arrow[r, "p"] \arrow[d, "\partial"'] & G \arrow[d, "\partial'"] \\
H_0 \arrow[r, "p_0"'] & G_0
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
H \arrow[r, "p"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\text{final}"] & G \arrow[r, equal] \arrow[d, "\widetilde{\partial}"'] \arrow[dr, phantom, "\text{d.f.}"] & G \arrow[d, "\partial'"] \\
H_0 \arrow[r, "\widetilde{p}_0"'] & \widetilde{G}_0 \arrow[r, "g_0"'] & G_0
\end{tikzcd}
$$

The push forward and final functors

. . . also it gives a common framework where to place two *apparently unrelated* situations:

$$\begin{tikzcd}[column sep=small] & \textbf{push forward of crossed modules} & \\ \textbf{final functors} \arrow[ur, Rightarrow] & & \textbf{push forward of extensions} \arrow[ul, Rightarrow] \end{tikzcd}$$

The push forward and final functors

Combining the theorem with the proposition above,
one gets the following characterization:

Corollary A morphism of crossed modules is final iff it induces an isomorphism between the cokernels and a regular epimorphism between the kernels.

It is interesting to translate this for internal groupoids.
Recall that for a groupoid $\mathbb{G}$, we define

- $\pi_0(\mathbb{G}) = \mathrm{Coeq}(d, c) = \mathrm{Coker}(\partial)$
- $\pi_1(\mathbb{G}) = \mathrm{Ker}(c) \cap \mathrm{Ker}(d) = \mathrm{Ker}(\partial)$

Corollary An internal functor is final iff it induces an isomorphism between the π_0's and a regular epimorphism between the π_1's.

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$$
\begin{tikzcd}
\mathrm{Ker}(\partial) \arrow[r, two heads] \arrow[d] & \mathrm{Ker}(\widetilde{\partial}) \arrow[r, tail] \arrow[d] & \mathrm{Ker}(\partial') \arrow[d] \\
H \arrow[r, "p"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\text{final}"] & G \arrow[r, equal] \arrow[d, "\widetilde{\partial}"'] \arrow[dr, phantom, "\text{d.f.}"] & G \arrow[d, "\partial'"] \\
H_0 \arrow[r, "\widetilde{p}_0"'] \arrow[d] & \widetilde{G}_0 \arrow[r, "g_0"'] \arrow[d] & G_0 \arrow[d] \\
\mathrm{Coker}(\partial) \arrow[r] & \mathrm{Coker}(\widetilde{\partial}) \arrow[r, equal] & \mathrm{Coker}(\partial')
\end{tikzcd}
$$

2. The push forward in the non-pointed context

The Barr-exact, Mal'tsev setting

- The final / discrete fibration factorization system for internal groupoids has shown to be a useful tool for developing intrinsic categorical algebra in Barr-exact, Mal'tsev categories.
- One may ask if the push forward construction has a corresponding non-pointed version in such a context.
- The answer is yes.

Notice. Throughout this section we refer to internal groupoids, but keep in mind that, since the base category is Mal'tsev, all internal categories are groupoids, so that all could be stated in terms of internal categories.

The Barr-exact, Mal'tsev setting

Theorem Consider $\mathbb{H} = \left\{ \begin{tikzcd} H_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & H_0 \arrow[l, "e"' description] \end{tikzcd} \right\}$ a groupoid

in $\mathcal{C}$ Barr-exact Mal'tsev, together with a morphism $\widehat{p}$ of split epis

$$\begin{tikzcd} H_1 \arrow[rr, "\widehat{p}"] \arrow[dr, shift right, "c"'] & & T \arrow[dl, shift left, "t"] \\ & H_0 \arrow[ul, shift right, "e"'] \arrow[ur, shift left, "s"] & \end{tikzcd}$$

such that

$$d^*((t,s)) \simeq \widehat{p}^*((t_2, \Delta)) \qquad \begin{tikzcd} T \arrow[d, shift right, "t"'] & P \arrow[l] \arrow[r] \arrow[d, shift right, "\bar{t}"'] \arrow[dl, phantom, "\llcorner" very near start] \arrow[dr, phantom, "\lrcorner" very near start] & R_t \arrow[d, shift right, "t_2"'] \\ H_0 \arrow[u, shift right, "s"'] & H_1 \arrow[l, "d"] \arrow[r, "\widehat{p}"'] \arrow[u, shift right, "\bar{s}"'] & T \arrow[u, shift right, "\Delta"'] \end{tikzcd}$$

The Barr-exact, Mal'tsev setting

then there exists a groupoid $\widetilde{\mathbb{G}} = \{\ G_1 \overset{d}{\underset{c}{\substack{\longrightarrow \\ \xleftarrow{e} \\ \longrightarrow}}} G_0\ \}$ and a functor $\widetilde{P} = (\widetilde{p}_1, \widetilde{p}_0) \colon \mathbb{G} \to \widetilde{\mathbb{G}}$ such that:

$$\widetilde{p}_0^*((c,e)) = (t,s) \qquad \begin{tikzcd} T \arrow[r] \arrow[d, shift right, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{G}_1 \arrow[d, shift right, "c"'] \\ H_0 \arrow[u, shift right, "s"'] \arrow[r, "\widetilde{p}_0"'] & \widetilde{G}_0 \arrow[u, shift right, "e"'] \end{tikzcd}$$

The Barr-exact, Mal'tsev setting

Moreover, for any other $P = (p_1, p_0)\colon \mathbb{H} \to \mathbb{G}$ such that

$$p_0^*((c,e)) = (t,s) \qquad \begin{tikzcd} T \arrow[r, "\overline{p}_0"] \arrow[d, shift right, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & G_1 \arrow[d, shift right, "c"'] \\ H_0 \arrow[u, shift right, "s"'] \arrow[r, "p_0"'] & G_0 \arrow[u, shift right, "e"'] \end{tikzcd} \qquad \text{and } \overline{p}_0\widehat{p} = p_1,$$

then there exists a unique functor (a discrete fibration) $\widetilde{G}\colon \widetilde{\mathbb{G}} \to \mathbb{G}$ such that

$$\widetilde{G}\widetilde{P} = P \qquad \begin{tikzcd} \mathbb{H} \arrow[r, "\widetilde{P}"] \arrow[dr, "P"'] & \widetilde{\mathbb{G}} \arrow[d, "\widetilde{G}"] \\ & \mathbb{G} \end{tikzcd}$$

The Barr-exact, Mal'tsev setting

We can replicate the main result for crossed module in the non-pointed context.

Theorem Let $\mathcal{C}$ be Barr-exact and Mal'tsev, and $P = (p_1, p_0) \colon \mathbb{H} \to \mathbb{G}$ a morphism in **Gpd**$(\mathcal{C})$.
T.F.A.E.

- P is the push forward of ∂ along p_1, w.r.t. $p^*((c,e))$ + the comparison map $\widehat{p}$
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The full treatment of the non-pointed case is currently under investigation...

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3. Some applications to butterflies

Butterflies

Finally we recollect some applications of the push forward construction.

Butterflies

Internal butterflies (Abbad, Vitale, Mantovani and M.) intrinsically describe weak morphisms of crossed modules.

A butterfly $E \colon \mathbb{H} \longrightarrow \mathbb{G}$ is a diagram satisfying four axioms:

$$\begin{tikzcd} H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \\ & E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & \\ H_0 & & G_0 \end{tikzcd}$$

1. (κ, ρ) is a complex
2. (ι, σ) is short exact

3. and 4. left and right wings equivariance

Butterflies reduced composition

Butterflies are the normalization of a special kind of profunctor (fractors). Their composition is quite involved.
Things get simpler when we compose a butterfly with a morphism of crossed modules. This is done via a pullback for the composition on the left, by means of a push forward for the composition on the right.

$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[dl, "\iota"'] \arrow[dd, "\partial"] \arrow[r, "f"] & X \arrow[dd, "\partial"] \\
& E \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & & \\
H_0 & & G_0 \arrow[r, "f_0'"'] & X_0
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$$
\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G \arrow[rr, "f"] \arrow[dl, "\iota"'] \arrow[dd, dotted] \arrow[dr, phantom, "\text{p.f.}"] & & X \arrow[dl] \arrow[dd, "\partial"] \\
& E \arrow[rr] \arrow[dl, "\sigma"] \arrow[dr, "\rho"'] & & E \rtimes^G X \arrow[dr] \arrow[dlll, dotted] & \\
H_0 & & G_0 \arrow[rr, "f'_0"'] & & X_0
\end{tikzcd}
$$

Butterflies reduced composition

Butterflies are the normalization of a special kind of profunctor (fractors). Their composition is quite involved.
Things get simpler when we compose a butterfly with a morphism of crossed modules. This is done via a pullback for the composition on the left, by means of a push forward for the composition on the right.

$$
\begin{tikzcd}
H \arrow[dr, "\kappa"] \arrow[dd, "\partial"'] & & G \arrow[dl] \arrow[rr, "f"] \arrow[dd, dotted] \arrow[dr, phantom, "\text{p.f.}"] & & X \arrow[dl] \arrow[dd, "\partial"] \\
& E \arrow[rr] \arrow[dl] \arrow[dr, dotted] & & E \rtimes^G X \arrow[dlll] \arrow[dr] & \\
H_0 & & {} \arrow[rr, dotted] & & X_0
\end{tikzcd}
$$

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$$
\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr, dotted] \arrow[drrr] & & {} \arrow[rr, dotted] \arrow[dl, dotted] & & X \arrow[dl] \arrow[dd, "\partial"] \\
& {} \arrow[dl, dotted] \arrow[rr, dotted] \arrow[dr, dotted] & & E \rtimes^G X \arrow[dlll] \arrow[dr] & \\
H_0 & & {} \arrow[rr, dotted] & & X_0
\end{tikzcd}
$$

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$$\begin{tikzcd}
H \arrow[d, "\partial"'] \arrow[dr] & & X \arrow[dl] \arrow[d, "\partial"] \\
H_0 & E \rtimes^G X \arrow[l] \arrow[r] & X_0
\end{tikzcd}$$

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$$\begin{tikzcd}
H \arrow[dd, "\partial"'] \arrow[dr] & & X \arrow[dl] \arrow[dd, "\partial"] \\
& E \rtimes^G X \arrow[dl] \arrow[dr] & \\
H_0 & & X_0
\end{tikzcd}$$

Braided crossed modules

From (Joyal and Street, 1986) we know that a braided 2-group can be described as 2-group $\mathbb{G}$ for which the tensor product $\otimes_{\mathbb{G}} \colon \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ is a monoidal functor.
If $\mathbb{G}$ is strict, this corresponds to a a braided crossed module of groups in the sense of (Conducé 1983):
$\partial \colon G \to G_0$ is endowed with a map

$$\{\ \ ,\ \ \} \colon G_0 \times G_0 \dashrightarrow G$$

satisfying, for any x, y, z in G_0, and a, b in G,

1. $\{x, y + z\} = y \cdot \{x, z\} + \{x, y\}$
2. $\{x + y, z\} = \{x, z\} + x \cdot \{y, z\}$
3. $\partial\{x, y\} = [y, x]$
4. $\{\partial a, x\} = x \cdot a - a$
5. $\{y, \partial b\} = b - y \cdot b$

Braided crossed modules

Extending the approach of Joyal and Street, we get:
Definition - Proposition. A crossed module $\mathbb{G}$ is braided (corresponding to a braided groupoid) if(f) it is equipped with a butterfly P and two morphisms $s_1, s_2 \colon G_0 \to P$

$$\begin{tikzcd}
G \times G \arrow[d, "\partial\times\partial"'] \arrow[dr, "\alpha"] & & G \arrow[dl, "\beta"'] \arrow[d, "\partial"] \\
G_0 \times G_0 & P \arrow[l, "\gamma"] \arrow[r, "\delta"'] & G_0
\end{tikzcd}$$

such that some diagrams commute.

What about Conducé approach?

Braided crossed modules

With any braided internal crossed module is associated a morphism $c\colon G_0 \otimes G_0 \to G$, where $G_0 \otimes G_0$ is the co-smash product in the sense of (Carboni Janelidze 2003).

$$\begin{tikzcd}
0 \arrow[r] & G_0 \otimes G_0 \arrow[r, "k"] \arrow[d, dashed, "c"'] & G_0 + G_0 \arrow[r, "\left\langle \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right\rangle"] \arrow[d, "{[s_1, s_2]}"] & G_0 \times G_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r, "\beta"'] & P \arrow[r, "\gamma"'] & G_0 \times G_0 \arrow[r] & 0
\end{tikzcd}$$

In fact β is the push forward of k along c.

Now, c is the internal version of Conducé's $\{\ ,\ \}$, so that it is possible to characterize braided crossed modules as those crossed modules ∂ endowed with a morphism $c\colon G_0 \otimes G_0 \to G$ that produces such a butterfly.

Braided crossed modules

The axioms are not-so-nice commutative diagrams (hope to get them simpler!). One of them gives the action:

$$\begin{tikzcd} G_0 \otimes G_0 \arrow[r, "k"] \arrow[d, "c"'] & G_0 + G_0 \arrow[d, "{[1,1]}"] \\ G \arrow[r, "\partial"'] & G_0 \end{tikzcd}$$

Braided crossed modules

$$\begin{tikzcd}
0 \arrow[r] & G_0 \otimes G_0 \arrow[r, "k"] \arrow[d, "c"'] & G_0 + G_0 \arrow[r, "\left\langle \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right\rangle"] \arrow[d, dashed] & G_0 \times G_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[r, dashed, "\beta"'] & P \arrow[r, dashed, "\gamma"'] & G_0 \times G_0 \arrow[r] & 0
\end{tikzcd}$$

What is remarkable here is that the canonical short exact sequence of the co-smash product encodes all the braidings over G_0, moreover the map c contains most of the cohomological information necessary to produce the extension (β, γ).

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