Probabilistic metric spaces as enriched categories

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Quantales

Definition

A quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$*, is a complete ordered set* V *equipped with an associative and commutative binary operation* $\otimes : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ *with neutral element* k *satisfying*

$$u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} (u \otimes v_i), \forall u, v_i \in \mathsf{V}, \forall i \in I.$$

Therefore $u \otimes - : \mathsf{V} \to \mathsf{V} \dashv hom(u, -)$.

Quantales

Example

- $2 = \{\mathsf{false}, \mathsf{true}\}$ *is a quantale with tensor* $\otimes = \&$ *and* $k = \mathsf{true}$. *More general, every frame is a quantale with* $\otimes = \wedge$ *and* $k = \top$.

- $([0, \infty], \geq, +, 0)$ *is a quantale, and one has*

$$\hom(x, y) = y \ominus x := \max\{y - x, 0\}$$

with $y - \infty = 0$ *and* $\infty - x = \infty$ *for* $x, y \in [0, \infty]$, $x \neq \infty$.

- $([0, 1], \leq, \times, 1)$ *is a quantale. The right adjoint is given here by "division"* $\hom(x, y) = y \oslash x := \min\{\frac{y}{x}, 1\}$ *for* $x \neq 0$ *and* $y \oslash 0 = 1$.

Quantales

Example

The set

$$\Delta = \{f : [0, +\infty] \to [0, 1], f \text{ monotone and } f(x) = \bigvee_{y<x} f(y)\}$$

is a quantale considering in Δ*:*

1. $f \leq g$ *iff* $f(x) \leq g(x), \forall x \in [0, +\infty]$;
2. $f \otimes g(x) = \bigvee_{y+z \leq x} (f(y) * g(z))$;
3. $f_{0,1}$ *is the neutral element.*

We call $f \in \Delta$ *finite if* $f(\infty) = 1$*.*

Morphism of quantales

Definition

Given also a quantale $\mathsf{W} = (\mathsf{W}, \oplus, l)$, *a monotone map* $F : \mathsf{V} \to \mathsf{W}$, *is a morphism of quantales whenever for all* $u, v, v_i \in \mathsf{V}$ *and* $i \in I$

$$\bigvee_{i \in I} F(v_i) = F(\bigvee_{i \in I} v_i), \qquad F(u) \oplus F(v) = F(u \otimes v), \qquad l = F(k),$$

For many applications it is enough to have inequalities above:

Definition

We say that F *is a lax morphism of quantales if, for all* $u, v \in \mathsf{V}$,

$$F(u) \oplus F(v) \leq F(u \otimes v), \qquad\qquad l \leq F(k).$$

Morphism of quantales

Example

The morphism of quantales

$$\begin{aligned} I : 2 &\to [0,\infty] \\ t &\mapsto 0 \\ f &\mapsto \infty \end{aligned}$$

has a left and a right adjoint given, respectively, by

$$O : [0,\infty] \to 2, \qquad x \mapsto \begin{cases} t & \text{if } x < \infty \\ f & \text{if } x = \infty \end{cases}$$

$$P : [0,\infty] \to 2 \qquad x \mapsto \begin{cases} t & \text{if } x = 0 \\ f & \text{if } x > 0 \end{cases}$$

Here $O : [0,\infty] \to 2$ *is a morphism of quantales as well, but* $P : [0,\infty] \to 2$ *is only a lax morphism of quantales.*

Morphism of quantales

Example

The quantale $[0,\infty]$ *embeds canonically into* Δ *via the morphism*

$$\begin{aligned} I_\infty : [0,\infty] &\to \Delta \\ x &\mapsto f_{x,1} \end{aligned}$$

I_∞ *has right adjoint and left adjoint:*

$$\begin{aligned} P_\infty : \Delta &\to [0,\infty] \\ f &\mapsto \inf\{x \in [0,\infty] \mid f(x) = 1\} \end{aligned}$$

$$\begin{aligned} O_\infty : \Delta &\to [0,\infty] \\ f &\mapsto \sup\{x \in [0,\infty] \mid f(x) = 0\} \end{aligned}$$

P_∞ *is only a lax morphism of quantales.*

V-Cat

For a quantale $(\mathsf{V}, \leq, \otimes, k)$, the category V-Cat has

- Objects: V-categories (X, a) st $a : X \times X \to \mathsf{V}$ and
  - $k \leq a(x, x)$;
  - $a(x, y) \otimes a(y, z) \leq a(x, z)$.
- Morphisms: V-functors - maps $f : (X, a) \to (Y, b)$ st

$$a(x, y) \leq b(f(x), f(y))$$

.

The quantale V gives rise to the V-category $\mathsf{V} = (\mathsf{V}, \hom)$.
Any V-category (X, a) is an ordered set.

V-Cat

Example

1. *A 2-category is just a set equipped with a reflexive and transitive relation, and a 2-functor is a monotone map. Hence,* 2-Cat $\simeq$ Ord.

2. *A* $[0,\infty]$*-category structure is a distance function* $a : X \times X \to [0,\infty]$ *which satisfies the conditions*

$$0 \geqslant a(x,x) \qquad \text{and} \qquad a(x,y) + a(y,z) \geqslant a(x,z),$$

for all $x, y, z \in X$*; and a* $[0,\infty]$*-functor is a non-expansive map. Hence,* $[0,\infty]$-Cat $\simeq$ Met.

V-Cat

Example

A probabilistic metric space $(X, a, *)$ *is a separated, symmetric and finitary* Δ*-category* (X, a)*.*

- $a(x, y) : [0, \infty] \to [0, 1]$ *satisfies for all* $x, y, z \in X$ *and* $t, s \in [0, \infty]$*:*
  1. $a(x, y) : [0, \infty] \to [0, 1]$ *is left continuous;*
  2. $\forall t > 0, a(x, x)(t) = 1$;
  3. $a(x, y)(t) * a(y, z)(s) \leq a(x, z)(t + s)$;
  4. $\forall t > 0, a(x, y)(t) = 1 \Rightarrow x = y$;
  5. $a(x, y) = a(y, x)$
  6. $a(x, y)(\infty) = 1$

We will use the term "probabilistic metric space" as a synonym for Δ*-category; then* ProbMet $\simeq$ Δ-Cat.

V-Cat

A lax morphism of quantales $F : \mathsf{V} \to \mathsf{W}$ induces a functor $F : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{W}\text{-}\mathsf{Cat}$ st

- $F(X, a) = (X, Fa)$ with $Fa = F \cdot a$:

$$X \times X \xrightarrow{a} \mathsf{V} \xrightarrow{F} \mathsf{W};$$

- $Ff := f : (X, F \cdot a) \to (Y, F \cdot b)$ for a V-functor $f : (X, a) \to (Y, b)$.

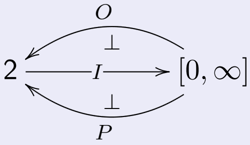
If $G : \mathsf{W} \to \mathsf{V}$ is also a lax morphism of quantales and $F \dashv G$ then the induced functors are also adjoint.

In particular, if $F = G^{-1}$, then $\mathsf{V}\text{-}\mathsf{Cat} \simeq \mathsf{W}\text{-}\mathsf{Cat}$.

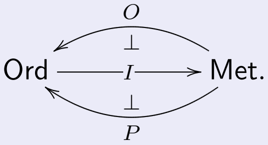
V-Cat

Example

We have seen that



Therefore, one obtains adjunctions between the induced functors:



V-Cat

Example

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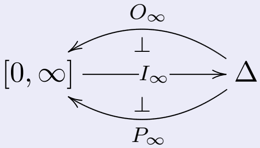
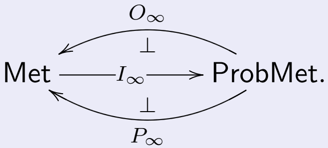


Therefore, one obtains the chain of functors



V-Dist

Definition

A V*-distributor* $\varphi : (X,a) \longrightarrow\!\!\!\!\!\circ\ (Y,b)$ *is a map* $\varphi : X \times Y \to$ V *such that*

$$\varphi \cdot a \leq \varphi \text{ and } b \cdot \varphi \leq \varphi.$$

In the category V-Dist of V-categories and V-distributors we consider:

- For $\psi : (Y,b) \longrightarrow\!\!\!\!\!\circ\ (Z,c)$:

$$\psi \cdot \varphi(x,z) = \bigvee_{y \in Y} \varphi(x,y) \otimes \psi(y,z),$$

- $a : (X,a) \longrightarrow\!\!\!\!\!\circ\ (X,a)$ is the identity on $X = (X,a)$.

Lemma

Let $\varphi, \varphi' : X \longrightarrow\!\!\!\!\!\circ\ Y$ *and* $\psi, \psi' : Y \longrightarrow\!\!\!\!\!\circ\ X$ *be* V*-distributors with* $\varphi \dashv \psi$, $\varphi' \dashv \psi'$, $\varphi \leq \varphi'$ *and* $\psi \leq \psi'$*. Then* $\varphi = \varphi'$ *and* $\psi = \psi'$*.*

V-Dist

There are two important functors:

$$(-)_* : \text{V-Cat} \to \text{V-Dist} \quad (-)^* : \text{V-Cat}^{\mathrm{op}} \to \text{V-Dist}.$$

that leave objects unchanged and for every V-functor $f : (X, a) \to (Y, b)$

$$f_* : (X, a) \multimap (Y, b) \quad f^* : (Y, b) \multimap (X, a),$$

st

$$f_*(x, y) = b(f(x), y) \quad f^*(y, x) = b(y, f(x)).$$

Furthermore,

$$f_* \dashv f^*$$

V-Dist

For every morphism of quantales $F : \mathsf{V} \to \mathsf{W}$:

$$\begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Dist} & \xrightarrow{F} & \mathsf{W}\text{-}\mathsf{Dist} \\ {\scriptstyle (-)_*}\uparrow & & \uparrow{\scriptstyle (-)_*} \\ \mathsf{V}\text{-}\mathsf{Cat} & \xrightarrow[F]{} & \mathsf{W}\text{-}\mathsf{Cat} \end{array} \qquad \begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Dist} & \xrightarrow{F} & \mathsf{W}\text{-}\mathsf{Dist} \\ {\scriptstyle (-)^*}\uparrow & & \uparrow{\scriptstyle (-)^*} \\ \mathsf{V}\text{-}\mathsf{Cat}^{\mathrm{op}} & \xrightarrow[F^{\mathrm{op}}]{} & \mathsf{W}\text{-}\mathsf{Cat}^{\mathrm{op}} \end{array}$$

For a V-distributor $\varphi : (X, a) \multimap (Y, b)$, $F\varphi = F \cdot \varphi$.
F is even locally monotone:

$$\varphi \leq \varphi' \Longrightarrow F\varphi \leq F\varphi';$$

therefore:

$$\varphi \dashv \psi \text{ in V-Dist} \Rightarrow F\varphi \dashv F\psi \text{ in W-Dist}$$

Cauchy Complete V-categories

Definition

A V*-categorie* (X, a) *is Cauchy complete if any left adjoint* V*-distributor* $\varphi : E \longrightarrow\!\!\!\!\!\circ\, X$ *is representable:* $\varphi = x_*$*, for some* $x \in X$*.*

Hence,

X is Cauchy complete

- iff $(-)_* : X \to \{\varphi : E \longrightarrow\!\!\!\!\!\circ\, X \mid \varphi \text{ is l.a.}\}$ is surjective.
- iff $(-)^* : X \to \{\psi : X \longrightarrow\!\!\!\!\!\circ\, E \mid \psi \text{ is r.a.}\}$ is surjective.

Cauchy Complete V-categories

Let $F : \mathsf{V} \to \mathsf{W}$ be a morphism of quantales and X be a V-category.

$$
\begin{array}{ccc}
\{\varphi : E \mathrel{\circ\!\!\!\rightarrow} X \mid \varphi \text{ is l.a.}\} & \xrightarrow{\quad\Phi\quad} & \{\varphi' : FE \mathrel{\circ\!\!\!\rightarrow} FX \mid \varphi' \text{ is l.a.}\} \\
{\scriptstyle (-)_*}\nwarrow & & \nearrow{\scriptstyle (-)_*} \\
& |X| = |FX|, &
\end{array}
$$

Proposition

1. *FX is Cauchy complete and Φ is injective $\Rightarrow$ X is Cauchy complete.*
2. *X is Cauchy complete and Φ is surjective $\Rightarrow$ FX is Cauchy complete.*

Cauchy Complete V-categories

To obtain surjectivity of Φ, we assume that

- $F : \mathsf{V} \to \mathsf{W}$ is injective (then Φ is injective for every V-category X);
- there is a morphism of quantales $G : \mathsf{W} \to \mathsf{V}$ st $F \dashv G$.

Hence,

$$(\varphi' : E \,{\to\!\!\!\!\circ}\, FX) \dashv (\psi' : FX \,{\to\!\!\!\!\circ}\, E) \text{ in W-Dist}$$

gives

$$(G\varphi' : E \,{\to\!\!\!\!\circ}\, GFX) \dashv (G\psi' : GFX \,{\to\!\!\!\!\circ}\, E) \text{ in V-Dist.}$$

Since $GF = 1_{\mathsf{V}}$

$$(G\varphi' : E \,{\to\!\!\!\!\circ}\, X) \dashv (G\psi' : X \,{\to\!\!\!\!\circ}\, E) \text{ in V-Dist.}$$

and

$$(FG\varphi' : E \,{\to\!\!\!\!\circ}\, FX) \dashv (FG\psi' : FX \,{\to\!\!\!\!\circ}\, E) \text{ in W-Dist.}$$

with

$$FG\varphi' \leq \varphi' \text{ and } FG\psi' \leq \psi'$$

In these conditions we conclude that $FG\varphi' = \varphi'$.

Cauchy Complete V-categories

We also have surjectivity of Φ if $G \dashv F$, since:

- 1_X is a V-functor of type $\gamma : GFX \to X$,
- $(G\varphi' : E \multimap GFX) \dashv (G\psi' : GFX \multimap E)$ can be composed with $\gamma_* \dashv \gamma^*$ to yield

$$(\gamma_* \cdot G\varphi' : E \multimap X) \dashv (G\psi' \cdot \gamma^* : X \multimap E).$$

- $F\gamma$ is the identity on FX since $FGF = F$,
- Hence $\Phi(\gamma_* \cdot G\varphi') = \varphi'$.

Cauchy Complete V-categories

Corollary

Let $F : \mathsf{V} \to \mathsf{W}$ and $G : \mathsf{W} \to \mathsf{V}$ be morphisms of quantales and assume that either $G \dashv F$ or that $F \dashv G$ and F is injective. Then FX is Cauchy complete provided that X is Cauchy complete.

Example

Since $O_\infty \dashv I_\infty$, a metric space X is Cauchy complete in Met *if and only if $I_\infty X$ is Cauchy complete in* ProbMet.

Topology in a V-category

To every metric on a set X one associates a topology by putting, for all $M \subseteq X$ and $x \in X$:

$$x \in \overline{M} :\Leftrightarrow \exists \text{ (Cauchy) sequence } (x_n)_{n\in\mathbb{N}} \text{ in } M \text{ st } (x_n)_{n\in\mathbb{N}} \to x,$$

In the language of V-distributors:

$$\begin{aligned} x \in \overline{M} : &\Leftrightarrow x \text{ represents an adjoint pair of V-distributors on } M \\ &\Leftrightarrow (i^* \cdot x_* : E \mathrel{-\!\!\circ\!\!\to} M) \dashv (x^* \cdot i_*) : M \mathrel{-\!\!\circ\!\!\to} E \end{aligned}$$

where we consider M as a sub-V-category of X and $i : M \to X$ denotes the inclusion V-functor.

This latter formulation defines a closure operator for any V-category X.

Topology in a V-category

We recall :

Proposition

Let $X = (X, a)$ *be a* V*-category,* $M \subseteq X$ *and* $x \in X$*. Then*

$$x \in \overline{M} \Leftrightarrow k \leq \bigvee_{y \in M} a(x, y) \otimes a(y, x).$$

By the proposition above, for $x, x' \in \overline{M}$ one has

$$a(x, x') = \bigvee_{y \in M} a(x, y) \otimes a(y, x').$$

Topology in a V-category

Proposition

Let $f : X \to Y$ *be a* V*-functor,* $M, M' \subseteq X$ *and* $N \subseteq Y$*. Then*

1. $M \subseteq \overline{M}$,
2. $M \subseteq M'$ *implies* $\overline{M} \subseteq \overline{M'}$,
3. $\overline{\varnothing} = \varnothing$ *and* $\overline{\overline{M}} = \overline{M}$,
4. $f(\overline{M}) \subseteq \overline{f(M)}$ *and* $\overline{f^{-1}(N)} \subseteq f^{-1}(\overline{N})$,
5. $\overline{M \cup M'} = \overline{M} \cup \overline{M'}$ *provided that* $k \leq u \vee v$ *implies* $k \leq u$ *or* $k \leq v$ *for all* $u, v \in$ V*.*

Furthermore, $\overline{(-)}$ *is hereditary, that is, for* $M \subseteq Z \subseteq X$*, where* Z *is a sub-*V*-category of* X*:*

$$\overline{M}_{\text{in } Z} = \overline{M}_{\text{in } X} \cap Z.$$

Topology in a V-category

One has the expected results linking closed subsets with Cauchy completeness:

- every closed subset of a Cauchy complete V-category is Cauchy complete;
- every Cauchy complete sub-V-category of a separated V-category is closed.

The inclusion V-functor $i : M \to X$ is fully dense (i.e. $i_* \cdot i^* = a$ where $X = (X, a)$) if and only if $\overline{M} = X$.

Topology in a V-category

Example: $y_X : X \to [X^{\mathrm{op}}, \mathsf{V}]$, since

$$\overline{y_X(X)} = \widetilde{X} = \{\psi : X \multimapdotboth 1 \mid \psi \text{ is right adjoint}\}.$$

Hence,

- $\widetilde{X}$ is Cauchy complete;
- $y_X : X \to \tilde{X}$ is (fully faithful and) fully dense;
- $(y_X)_* : X \multimapdotboth \tilde{X}$ is an isomorphism in V-Dist with inverse $y_X^* : \tilde{X} \multimapdotboth X$.

Then:

For every V-functor $f : X \to Y$ where Y is separated and Cauchy complete, there exists a unique V-functor $g : \tilde{X} \to Y$ with $g \cdot y_X = f$.

g can be taken as the V-functor $\tilde{X} \to Y$ st $g_* = f_* \cdot y_X^*$, it exists since Y is Cauchy complete and it is unique since Y is separated.

Topology in a V-category

Proposition

A V*-category* X *is Cauchy complete if and only if* X *is injective with respect to fully faithful and fully dense* V*-functors.*

Lemma

Let F : W $\to$ V *be a morphism of quantales. Then* F : W-Cat $\to$ V-Cat *sends fully faithful and fully dense* W*-functors to fully faithful and fully dense* V*-functors.*

Example

P_∞ : ProbMet $\to$ Met *preserves Cauchy completeness.*