Probabilistic metric spaces as enriched categories

Carla Reis

Polytechnic Institute of Coimbra
Centro de Investigação e Desenvolvimento em Matemática e Aplicações, UA

Workshop on Category Theory, Universidade de Coimbra, July 2012
Quantales

Definition

A quantale \( V = (V, \otimes, k) \), is a complete ordered set \( V \) equipped with an associative and commutative binary operation \( \otimes : V \times V \to V \) with neutral element \( k \) satisfying

\[
u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} (u \otimes v_i), \forall u, v_i \in V, \forall i \in I.\]

Therefore \( u \otimes - : V \to V \dashv \text{hom}(u, -) \).
Quantales

Example

- $2 = \{ \text{false, true} \}$ is a quantale with tensor $\otimes = \&$ and $k = \text{true}$. More general, every frame is a quantale with $\otimes = \wedge$ and $k = \top$.

- $([0, \infty], \geq, +, 0)$ is a quantale, and one has

$$\text{hom}(x, y) = y \ominus x := \max\{y - x, 0\}$$

with $y - \infty = 0$ and $\infty - x = \infty$ for $x, y \in [0, \infty], x \neq \infty$.

- $([0, 1], \leq, \times, 1)$ is a quantale. The right adjoint is given here by “division” $\text{hom}(x, y) = y \odot x := \min\{\frac{y}{x}, 1\}$ for $x \neq 0$ and $y \odot 0 = 1$.
Quantales

Example

The set

$$\Delta = \{ f : [0, +\infty] \to [0, 1], f \text{ monotone and } f(x) = \bigvee_{y < x} f(y) \}$$

is a quantale considering in $\Delta$:

1. $f \leq g$ iff $f(x) \leq g(x)$, $\forall x \in [0, +\infty]$;
2. $f \otimes g(x) = \bigvee_{y+z \leq x} (f(y) \ast g(z))$;
3. $f_{0,1}$ is the neutral element.

We call $f \in \Delta$ finite if $f(\infty) = 1$. 
Morphism of quantales

**Definition**

Given also a quantale \( W = (W, \oplus, l) \), a monotone map \( F : V \to W \), is a morphism of quantales whenever for all \( u, v, v_i \in V \) and \( i \in I \)

\[
\bigvee_{i \in I} F(v_i) = F\left( \bigvee_{i \in I} v_i \right), \quad F(u) \oplus F(v) = F(u \otimes v), \quad l = F(k),
\]

For many applications it is enough to have inequalities above:

**Definition**

We say that \( F \) is a lax morphism of quantales if, for all \( u, v \in V \),

\[
F(u) \oplus F(v) \leq F(u \otimes v), \quad l \leq F(k).
\]
Example

The morphism of quantales

\[ I : 2 \to [0, \infty] \]

\[ t \mapsto 0 \]
\[ f \mapsto \infty \]

has a left and a right adjoint given, respectively, by

\[ O : [0, \infty] \to 2, \quad P : [0, \infty] \to 2 \]

\[ x \mapsto \begin{cases} t & \text{if } x < \infty \\ f & \text{if } x = \infty \end{cases} \]
\[ x \mapsto \begin{cases} t & \text{if } x = 0 \\ f & \text{if } x > 0 \end{cases} \]

Here \( O : [0, \infty] \to 2 \) is a morphism of quantales as well, but \( P : [0, \infty] \to 2 \) is only a lax morphism of quantales.
Morphism of quantales

Example

The quantale $[0, \infty]$ embeds canonically into $\Delta$ via the morphism

$$I_\infty : [0, \infty] \to \Delta$$

$$x \mapsto f_{x,1}$$

$I_\infty$ has right adjoint and left adjoint:

$$P_\infty : \Delta \to [0, \infty]$$

$$f \mapsto \inf \{x \in [0, \infty] \mid f(x) = 1\}$$

$$O_\infty : \Delta \to [0, \infty]$$

$$f \mapsto \sup \{x \in [0, \infty] \mid f(x) = 0\}$$

$P_\infty$ is only a lax morphism of quantales.
V-Cat

For a quantale \((V, \leq, \otimes, k)\), the category \(V\)-Cat has

- **Objects**: \(V\)-categories \((X, a)\) st \(a : X \times X \to V\) and
  - \(k \leq a(x, x)\);
  - \(a(x, y) \otimes a(y, z) \leq a(x, z)\).

- **Morphisms**: \(V\)-functors - maps \(f : (X, a) \to (Y, b)\) st
  \[
a(x, y) \leq b(f(x), f(y))
  \]

The quantale \(V\) gives rise to the \(V\)-category \(V = (V, \text{hom})\). Any \(V\)-category \((X, a)\) is an ordered set.
A 2-category is just a set equipped with a reflexive and transitive relation, and a 2-functor is a monotone map. Hence, $2\text{-Cat} \cong \text{Ord}$. 

A $[0, \infty]$-category structure is a distance function $a : X \times X \to [0, \infty]$ which satisfies the conditions:

$$0 \geq a(x, x) \quad \text{and} \quad a(x, y) + a(y, z) \geq a(x, z),$$

for all $x, y, z \in X$; and a $[0, \infty]$-functor is a non-expansive map. Hence, $[0, \infty]$-Cat $\cong \text{Met}$. 
A probabilistic metric space \((X, a, \ast)\) is a separated, symmetric and finitary \(\Delta\)-category \((X, a)\).

- \(a(x, y) : [0, \infty] \rightarrow [0, 1]\) satisfies for all \(x, y, z \in X\) and \(t, s \in [0, \infty]\):
  1. \(a(x, y) : [0, \infty] \rightarrow [0, 1]\) is left continuous;
  2. \(\forall t > 0, a(x, x)(t) = 1\);
  3. \(a(x, y)(t) \ast a(y, z)(s) \leq a(x, z)(t + s)\);
  4. \(\forall t > 0, a(x, y)(t) = 1 \Rightarrow x = y\);
  5. \(a(x, y) = a(y, x)\)
  6. \(a(x, y)(\infty) = 1\)

We will use the term “probabilistic metric space” as a synonym for \(\Delta\)-category; then \(\text{ProbMet} \simeq \Delta\text{-Cat}\).
A lax morphism of quantales $F : V \to W$ induces a functor $F : V\text{-Cat} \to W\text{-Cat}$ st

1. $F(X, a) = (X, Fa)$ with $Fa = F \cdot a$:

\[ X \times X \xrightarrow{a} V \xrightarrow{F} W; \]

2. $Ff := f : (X, F \cdot a) \to (Y, F \cdot b)$ for a $V$-functor $f : (X, a) \to (Y, b)$.

If $G : W \to V$ is also a lax morphism of quantales and $F \dashv G$ then the induced functors are also adjoint.

In particular, if $F = G^{-1}$, then $V\text{-Cat} \simeq W\text{-Cat}$.
We have seen that

\[ 2 \xrightarrow{I} [0, \infty] \xleftarrow{P} \]

Therefore, one obtains adjunctions between the induced functors:

\[ \text{Ord} \xleftrightarrow{I} \text{Met.} \]
Example

We have seen that

\[
\begin{array}{c}
O_\infty \\
\downarrow \quad \downarrow \\
[0, \infty] & I_\infty & \rightarrow & \Delta \\
\downarrow \quad \downarrow \\
P_\infty
\end{array}
\]

Therefore, one obtains the chain of functors

\[
\begin{array}{c}
O_\infty \\
\downarrow \\
\downarrow \\
Met & I_\infty & \rightarrow & \text{ProbMet.} \\
\downarrow \\
P_\infty
\end{array}
\]
**V-Dist**

**Definition**

A **V-distributor** $\varphi : (X, a) \to (Y, b)$ is a map $\varphi : X \times Y \to V$ such that

$$\varphi \cdot a \leq \varphi \text{ and } b \cdot \varphi \leq \varphi.$$ 

In the category V-Dist of V-categories and V-distributors we consider:

- For $\psi : (Y, b) \to (Z, c)$:

  $$\psi \cdot \varphi(x, z) = \bigvee_{y \in Y} \varphi(x, y) \otimes \psi(y, z),$$

- $a : (X, a) \to (X, a)$ is the identity on $X = (X, a)$.

**Lemma**

Let $\varphi, \varphi' : X \to Y$ and $\psi, \psi' : Y \to X$ be V-distributors with $\varphi \dashv \psi$, $\varphi' \dashv \psi'$, $\varphi \leq \varphi'$ and $\psi \leq \psi'$. Then $\varphi = \varphi'$ and $\psi = \psi'$.
V-Dist

There are two important functors:

$(-)_* : V\text{-Cat} \to V\text{-Dist}$  \quad \quad $(-)^* : V\text{-Cat}^{\text{op}} \to V\text{-Dist}.$

that leave objects unchanged and for every V-functor $f : (X, a) \to (Y, b)$

$f_* : (X, a) \to \circ \to (Y, b) \quad f^* : (Y, b) \to \circ \to (X, a),$

st

$f_*(x, y) = b(f(x), y) \quad f^*(y, x) = b(y, f(x)).$

Furthermore,

$f_* \dashv f^*$
For every morphism of quantales $F : V \to W$:

For a V-distributor $\varphi : (X, a) \Rightarrow (Y, b)$, $F\varphi = F \cdot \varphi$.

$F$ is even locally monotone:

$$\varphi \leq \varphi' \implies F\varphi \leq F\varphi';$$

therefore:

$$\varphi \vdash \psi \text{ in } V\text{-Dist} \implies F\varphi \vdash F\psi \text{ in } W\text{-Dist}$$
Cauchy Complete $V$-categories

**Definition**

A $V$-categorie $(X, \alpha)$ is Cauchy complete if any left adjoint $V$-distributor $\varphi : E \rightarrow X$ is representable: $\varphi = x_\ast$, for some $x \in X$.

Hence, $X$ is Cauchy complete

- iff $(-)_\ast : X \rightarrow \{\varphi : E \rightarrow X \mid \varphi \text{ is l.a.}\}$ is surjective.
- iff $(-)^\ast : X \rightarrow \{\psi : X \rightarrow E \mid \psi \text{ is r.a.}\}$ is surjective.
Cauchy Complete V-categories

Let $F : V \to W$ be a morphism of quantales and $X$ be a V-category.

\[
\{ \varphi : E \to X \mid \varphi \text{ is l.a.} \} \xrightarrow{\Phi} \{ \varphi' : FE \to FX \mid \varphi' \text{ is l.a.} \}
\]

\[
|X| = |FX|,
\]

**Proposition**

1. $FX$ is Cauchy complete and $\Phi$ is injective $\Rightarrow$ $X$ is Cauchy complete.
2. $X$ is Cauchy complete and $\Phi$ is surjective $\Rightarrow$ $FX$ is Cauchy complete.
Cauchy Complete \( V \)-categories

To obtain surjectivity of \( \Phi \), we assume that

- \( F : V \rightarrow W \) is injective (then \( \Phi \) is injective for every \( V \)-category \( X \));
- there is a morphism of quantales \( G : W \rightarrow V \) st \( F \dashv G \).

Hence,

\[
(\varphi' : E \rightarrow FX) \vdash (\psi' : FX \rightarrow E) \text{ in } W\text{-Dist}
\]

gives

\[
(G\varphi' : E \rightarrow GFX) \vdash (G\psi' : GFX \rightarrow E) \text{ in } V\text{-Dist}.
\]

Since \( GF = 1_V \)

\[
(G\varphi' : E \rightarrow X) \vdash (G\psi' : X \rightarrow E) \text{ in } V\text{-Dist}.
\]

and

\[
(FG\varphi' : E \rightarrow FX) \vdash (FG\psi' : FX \rightarrow E) \text{ in } W\text{-Dist}.
\]

with

\[
FG\varphi' \leq \varphi' \text{ and } FG\psi' \leq \psi'
\]

In these conditions we conclude that \( FG\varphi' = \varphi' \).
Cauchy Complete V-categories

We also have surjectivity of $\Phi$ if $G \dashv F$, since:

- $1_X$ is a V-functor of type $\gamma : GFX \to X$,
- $(G\varphi' : E \to GFX) \dashv (G\psi' : GFX \to E)$ can be composed with $\gamma^* \dashv \gamma^*$ to yield

\[(\gamma^* \cdot G\varphi' : E \to X) \dashv (G\psi' \cdot \gamma^* : X \to E).\]

- $F\gamma$ is the identity on $FX$ since $FGF = F$,
- Hence $\Phi(\gamma^* \cdot G\varphi') = \varphi'$. 
**Corollary**

Let $F : V \to W$ and $G : W \to V$ be morphisms of quantales and assume that either $G \dashv F$ or that $F \dashv G$ and $F$ is injective. Then $FX$ is Cauchy complete provided that $X$ is Cauchy complete.

**Example**

Since $O_{\infty} \dashv I_{\infty}$, a metric space $X$ is Cauchy complete in Met if and only if $I_{\infty}X$ is Cauchy complete in ProbMet.
Topology in a V-category

To every metric on a set $X$ one associates a topology by putting, for all $M \subseteq X$ and $x \in X$:

$$x \in \overline{M} :\iff \exists \text{ (Cauchy) sequence } (x_n)_{n \in \mathbb{N}} \text{ in } M \text{ st } (x_n)_{n \in \mathbb{N}} \to x,$$

In the language of V-distributors:

$$x \in \overline{M} :\iff x \text{ represents an adjoint pair of V-distributors on } M$$

$$\iff (i^* \cdot x_* : E \to \rightarrow M) \dashv (x^* \cdot i_*) : M \to \rightarrow E$$

where we consider $M$ as a sub-V-category of $X$ and $i : M \to X$ denotes the inclusion V-functor.

This latter formulation defines a closure operator for any V-category $X$. 
We recall:

**Proposition**

Let \( X = (X, a) \) be a \( V \)-category, \( M \subseteq X \) and \( x \in X \). Then

\[
x \in \overline{M} \iff k \leq \bigvee_{y \in M} a(x, y) \otimes a(y, x).
\]

By the proposition above, for \( x, x' \in \overline{M} \) one has

\[
a(x, x') = \bigvee_{y \in M} a(x, y) \otimes a(y, x').
\]
Let $f : X \to Y$ be a $\mathbb{V}$-functor, $M, M' \subseteq X$ and $N \subseteq Y$. Then

1. $M \subseteq \overline{M}$,
2. $M \subseteq M'$ implies $\overline{M} \subseteq \overline{M}'$,
3. $\emptyset = \emptyset$ and $\overline{M} = \overline{M}$,
4. $f(\overline{M}) \subseteq f(\overline{M})$ and $f^{-1}(N) \subseteq f^{-1}(N)$,
5. $\overline{M \cup M'} = \overline{M} \cup \overline{M'}$ provided that $k \leq u \lor v$ implies $k \leq u$ or $k \leq v$ for all $u, v \in \mathbb{V}$.

Furthermore, $(\overline{-})$ is hereditary, that is, for $M \subseteq Z \subseteq X$, where $Z$ is a sub-$\mathbb{V}$-category of $X$: 

$$\overline{M}_{\text{in } Z} = \overline{M}_{\text{in } X} \cap Z.$$
One has the expected results linking closed subsets with Cauchy completeness:

- every closed subset of a Cauchy complete V-category is Cauchy complete;
- every Cauchy complete sub-V-category of a separated V-category is closed.

The inclusion V-functor \( i : M \to X \) is fully dense (i.e. \( i_* \cdot i^* = a \) where \( X = (X, a) \)) if and only if \( M = X \).
Topography in a **V**-category

**Example:** \( y_X : X \to [X^{\text{op}}, V] \), since

\[
y_X(X) = \tilde{X} = \{ \psi : X \to 1 \mid \psi \text{ is right adjoint} \}.
\]

Hence,

- \( \tilde{X} \) is Cauchy complete;
- \( y_X : X \to \tilde{X} \) is (fully faithful and) fully dense;
- \( (y_X)_* : X \to \tilde{X} \) is an isomorphism in V-Dist with inverse \( y_X^* : \tilde{X} \to X \).

Then:

For every \( V \)-functor \( f : X \to Y \) where \( Y \) is separated and Cauchy complete, there exists a unique \( V \)-functor \( g : \tilde{X} \to Y \) with \( g \cdot y_X = f \).

\( g \) can be taken as the \( V \)-functor \( \tilde{X} \to Y \) st \( g_* = f_* \cdot y_X^* \), it exists since \( Y \) is Cauchy complete and it is unique since \( Y \) is separated.
Proposition

A $V$-category $X$ is Cauchy complete if and only if $X$ is injective with respect to fully faithful and fully dense $V$-functors.

Lemma

Let $F : W \rightarrow V$ be a morphism of quantales. Then $F : W\text{-Cat} \rightarrow V\text{-Cat}$ sends fully faithful and fully dense $W$-functors to fully faithful and fully dense $V$-functors.

Example

$P_\infty : \text{ProbMet} \rightarrow \text{Met}$ preserves Cauchy completeness.