preamble

The cuboid lemma and Mal'tsev categories

Marino Gran

and

Diana Rodelo

drodelo@ualg.pt

Centre for Mathematics of the University of Coimbra

University of Algarve, Portugal

Contents

Preliminaries

Regular Mal'tsev categories

The Cuboid Lemma

The relative context

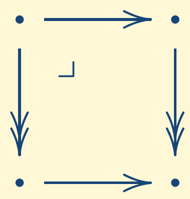
Contents

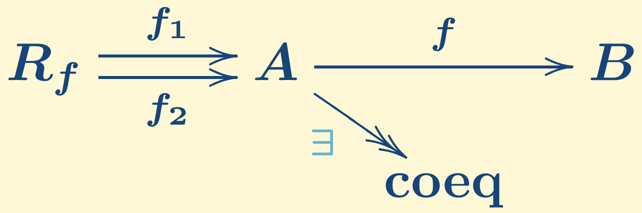
Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Preliminaries

Regular categories

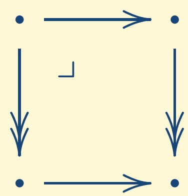
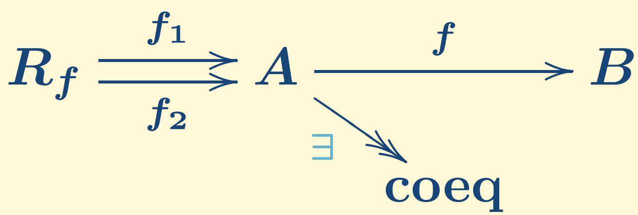
- $\mathcal{C}$ regular cat $=$ - lex

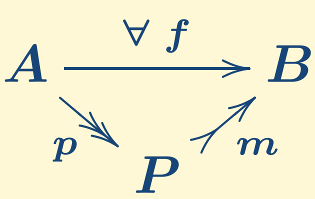
  - 

  - 

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

- $\mathcal{C}$ regular cat = - lex
  - 
  - 

- $\mathcal{C}$ regular $\Rightarrow$ 

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Regular categories

- $\mathcal{C}$ regular cat $=$
  - lex
  - $\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \lrcorner & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$
  - $R_f \overset{f_1}{\underset{f_2}{\rightrightarrows}} A \xrightarrow{f} B$, $A \to$ coeq ($\exists$)

- $\mathcal{C}$ regular $\Rightarrow$ $A \xrightarrow{\forall f} B$, $A \xrightarrow{p} P \xrightarrow{m} B$

- $\mathcal{C}$ regular

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Relations

- relation R from A to B $\rightsquigarrow$ $R \xrightarrow{\langle r_1, r_2 \rangle} A \times B$

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Relations

- relation R from A to B $\rightsquigarrow$ $R \overset{\langle r_1, r_2 \rangle}{\rightarrowtail} A \times B$

  opposite R° from B to A $\rightsquigarrow$ $R \overset{\langle r_2, r_1 \rangle}{\rightarrowtail} B \times A$

Contents
Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Relations

- relation R from A to B $\rightsquigarrow$ $R \overset{\langle r_1, r_2 \rangle}{\rightarrowtail} A \times B$

  opposite R° from B to A $\rightsquigarrow$ $R \overset{\langle r_2, r_1 \rangle}{\rightarrowtail} B \times A$

  map $A \xrightarrow{f} B$ $\rightsquigarrow$ $A \overset{f = \langle 1_A, f \rangle}{\rightarrowtail} A \times B$

  $\rightsquigarrow$ $A \overset{f^\circ = \langle f, 1_A \rangle}{\rightarrowtail} B \times A$

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

- relation R from A to B $\rightsquigarrow$ $R \xrightarrow{\langle r_1, r_2 \rangle} A \times B$

  opposite R° from B to A $\rightsquigarrow$ $R \xrightarrow{\langle r_2, r_1 \rangle} B \times A$

  map $A \xrightarrow{f} B$ $\rightsquigarrow$ $A \xrightarrow{f = \langle 1_A, f \rangle} A \times B$

  $\rightsquigarrow$ $A \xrightarrow{f^\circ = \langle f, 1_A \rangle} B \times A$

- $R \rightarrowtail A \times B$, $S \rightarrowtail B \times C$ $\rightsquigarrow$ $SR \rightarrowtail A \times C$

Contents
Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

- relation R from A to B $\rightsquigarrow$ $R \rightarrowtail^{\langle r_1, r_2 \rangle} A \times B$

  opposite R° from B to A $\rightsquigarrow$ $R \rightarrowtail^{\langle r_2, r_1 \rangle} B \times A$

  map $A \xrightarrow{f} B$ $\rightsquigarrow$ $A \rightarrowtail^{f = \langle 1_A, f \rangle} A \times B$

  $\rightsquigarrow$ $A \rightarrowtail^{f^\circ = \langle f, 1_A \rangle} B \times A$

- $R \rightarrowtail A \times B$, $S \rightarrowtail B \times C$ $\rightsquigarrow$ $SR \rightarrowtail A \times C$

- $R = r_2 r_1^\circ$ and $R_f = f^\circ f$ $(= f_2 f_1^\circ)$

Contents
Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

- relation R from A to B $\rightsquigarrow$ $R \rightarrowtail^{\langle r_1, r_2 \rangle} A \times B$

  opposite R° from B to A $\rightsquigarrow$ $R \rightarrowtail^{\langle r_2, r_1 \rangle} B \times A$

  map $A \xrightarrow{f} B$ $\rightsquigarrow$ $A \rightarrowtail^{f = \langle 1_A, f \rangle} A \times B$

  $\rightsquigarrow$ $A \rightarrowtail^{f^\circ = \langle f, 1_A \rangle} B \times A$

- $R \rightarrowtail A \times B$, $S \rightarrowtail B \times C$ $\rightsquigarrow$ $SR \rightarrowtail A \times C$

- $R = r_2 r_1^\circ$ and $R_f = f^\circ f$ $(= f_2 f_1^\circ)$

- $f f^\circ f = f$ and $f^\circ f f^\circ = f^\circ$ (difunctional)

Contents

Preliminaries

Regular categories

Relations

Equivalence relations

Regular Mal'tsev categories

The Cuboid Lemma

The relative context

- relation $\quad R$ from A to B $\quad\rightsquigarrow\quad R \xrightarrow{\langle r_1, r_2\rangle} A \times B$

  opposite $\quad R^\circ$ from B to A $\quad\rightsquigarrow\quad R \xrightarrow{\langle r_2, r_1\rangle} B \times A$

  map $\quad A \xrightarrow{f} B$ $\quad\rightsquigarrow\quad A \xrightarrow{f=\langle 1_A, f\rangle} A \times B$

  $\rightsquigarrow\quad A \xrightarrow{f^\circ=\langle f, 1_A\rangle} B \times A$

- $R \rightarrowtail A \times B$, $S \rightarrowtail B \times C$ $\quad\rightsquigarrow\quad SR \rightarrowtail A \times C$

- $R = r_2 r_1^\circ$ and $R_f = f^\circ f \ \ (= f_2 f_1^\circ\ )$

- $f f^\circ f = f$ and $f^\circ f f^\circ = f^\circ$ $\quad$ (difunctional)

- $f f^\circ = 1_B$ iff f regular epi

Contents
Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Contents
Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

- relation R from A to B $\rightsquigarrow$ $R \rightarrowtail^{\langle r_1, r_2 \rangle} A \times B$

  opposite R° from B to A $\rightsquigarrow$ $R \rightarrowtail^{\langle r_2, r_1 \rangle} B \times A$

  map $A \xrightarrow{f} B$ $\rightsquigarrow$ $A \rightarrowtail^{f = \langle 1_A, f \rangle} A \times B$

  $\rightsquigarrow$ $A \rightarrowtail^{f^\circ = \langle f, 1_A \rangle} B \times A$

- $R \rightarrowtail A \times B$, $S \rightarrowtail B \times C$ $\rightsquigarrow$ $SR \rightarrowtail A \times C$

- $R = r_2 r_1^\circ$ and $R_f = f^\circ f$ $(= f_2 f_1^\circ)$

- $f f^\circ f = f$ and $f^\circ f f^\circ = f^\circ$ (difunctional)

- $f f^\circ = 1_B$ iff f regular epi

relative

Equivalence relations

- $R \rightarrowtail A \times A$ is: reflexive $1_A \leq R$

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Equivalence relations

- $R \rightarrowtail A \times A$ is: reflexive $1_A \leq R$

  symmetric $R^{\circ} \leq R$

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Equivalence relations

- $R \rightarrowtail A \times A$ is: reflexive $1_A \leq R$

  symmetric $R^\circ \leq R$

  transitive $RR \leq R$

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Equivalence relations

- $R \rightarrowtail A \times A$ is: reflexive $1_A \leq R$

  symmetric $R^\circ \leq R$

  transitive $RR \leq R$

  $\Bigg]$ equivalence

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Equivalence relations

- $R \rightarrowtail A \times A$ is: reflexive $1_A \leq R$

  symmetric $R^{\circ} \leq R$

  transitive $RR \leq R$

  equivalence

- R_f effective equivalence relation

Contents
Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Equivalence relations

- $R \rightarrowtail A \times A$ is: reflexive $1_A \leq R$

  symmetric $R^\circ \leq R$

  transitive $RR \leq R$

  ] equivalence

- R_f effective equivalence relation

- exact fork $R_f \underset{f_2}{\overset{f_1}{\rightrightarrows}} A \overset{f}{\twoheadrightarrow} B$

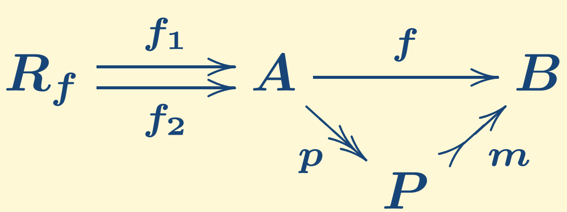
Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

- $R \rightarrowtail A \times A$ is: reflexive $1_A \leq R$; symmetric $R^{\circ} \leq R$; transitive $RR \leq R$ — equivalence

- R_f effective equivalence relation

- exact fork $R_f \overset{f_1}{\underset{f_2}{\rightrightarrows}} A \overset{f}{\twoheadrightarrow} B$

- 

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Equivalence relations

- $R \rightarrowtail A \times A$ is: reflexive $1_A \leq R$, symmetric $R^\circ \leq R$, transitive $RR \leq R$ — equivalence

- R_f effective equivalence relation

- exact fork $$R_f \underset{f_2}{\overset{f_1}{\rightrightarrows}} A \overset{f}{\twoheadrightarrow} B$$

- $$R_f \underset{f_2}{\overset{f_1}{\rightrightarrows}} A \overset{f}{\longrightarrow} B, \quad A \overset{p}{\twoheadrightarrow} P \overset{m}{\rightarrowtail} B \qquad \Rightarrow \quad R_f = R_p$$

Contents

Preliminaries
Regular categories
Relations
Equivalence relations
Regular Mal'tsev categories
The Cuboid Lemma
The relative context

Contents

Preliminaries

Regular Mal'tsev categories

Definition

Regular pushouts

Stability property

The Cuboid Lemma

The relative context

Regular Mal'tsev categories

Definition

- $\mathcal{C}$ Mal'tsev cat $=$ - lex

  - reflexive $=$ equivalence

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

Definition

- $\mathcal{C}$ Mal'tsev cat $=$ - lex

  - reflexive = equivalence

- Exs. - $\mathbf{Gp}$, $\mathbf{Alg}(\mathbb{T})$, $\mathbb{T}$ w/ group op.

  - quasi-groups, Heyting algebras
  - lex + additive
  - $(\mathbf{Topos})^{op}$
  - $\mathcal{C}$ Mal'tsev $\Rightarrow$ $\mathcal{C}/X$, $X/\mathcal{C}$, $\mathbf{Gp}(\mathcal{C})$, $\cdots$ Mal'tsev

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

- $\mathcal{C}$ Mal'tsev cat = - lex
  - reflexive = equivalence

- Exs. - $\mathbf{Gp}$, $\mathbf{Alg}(\mathbb{T})$, $\mathbb{T}$ w/ group op.
  - quasi-groups, Heyting algebras
  - lex + additive
  - $(\mathbf{Topos})^{op}$
  - $\mathcal{C}$ Mal'tsev $\Rightarrow$ $\mathcal{C}/X$, $X/\mathcal{C}$, $\mathbf{Gp}(\mathcal{C})$, $\cdots$ Mal'tsev

- Prop. [CLP, *Diagram chasing in Mal'cev cats*] $\mathcal{C}$ regular. TFAE:

  (a) $\mathcal{C}$ Mal'tsev cat

  (b) $R_f R_g = R_g R_f$

  ($RS = SR$ for equivalence relations)

- $$\begin{array}{ccc} C & \overset{c}{\twoheadrightarrow} & A \\ {\scriptstyle g}\downarrow\uparrow{\scriptstyle t} & (1) & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ D & \underset{d}{\twoheadrightarrow} & B, \end{array}$$

  $$g \cdot t = 1, \ \ f \cdot s = 1, \ \ c \cdot t = s \cdot d$$

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

Regular pushouts

- $$\begin{array}{ccc} C & \xrightarrow{c} & A \\ g\downarrow\uparrow t & (1) & f\downarrow\uparrow s \\ D & \xrightarrow[d]{} & B, \end{array}$$

  always a pushout

  $$g \cdot t = 1, \quad f \cdot s = 1, \quad c \cdot t = s \cdot d$$

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

Regular pushouts

- $$\begin{array}{ccc} C & \stackrel{c}{\twoheadrightarrow} & A \\ g \downarrow \uparrow t & (1) & f \downarrow \uparrow s \\ D & \underset{d}{\twoheadrightarrow} & B, \end{array}$$ always a pushout

  $g \cdot t = 1, \; f \cdot s = 1, \; c \cdot t = s \cdot d$

- regular po: $\langle g, c \rangle : C \twoheadrightarrow D \times_B A$ regular epi

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

Regular pushouts

- $$\begin{array}{ccc} C & \overset{c}{\twoheadrightarrow} & A \\ g\downarrow\uparrow t & (1) & f\downarrow\uparrow s \\ D & \underset{d}{\twoheadrightarrow} & B, \end{array}$$ always a pushout

  $$g \cdot t = 1,\;\; f \cdot s = 1,\;\; c \cdot t = s \cdot d$$

- regular po: $\langle g, c \rangle : C \twoheadrightarrow D \times_B A$ regular epi

  $$\Leftrightarrow \quad cg^\circ = f^\circ d \qquad (\; gc^\circ = d^\circ f \;)$$

Contents

Preliminaries

Regular Mal'tsev categories

Definition

Regular pushouts

Stability property

The Cuboid Lemma

The relative context

Regular pushouts

- $$\begin{array}{ccccc} R_c & \rightrightarrows & C & \overset{c}{\twoheadrightarrow} & A \\ \downarrow\uparrow & & g\downarrow\uparrow t & (1) & f\downarrow\uparrow s \\ R_d & \rightrightarrows & D & \underset{d}{\twoheadrightarrow} & B, \end{array}$$ always a pushout

  $$g \cdot t = 1,\ \ f \cdot s = 1,\ \ c \cdot t = s \cdot d$$

- regular po: $\langle g, c\rangle : C \twoheadrightarrow D \times_B A$ regular epi

  $$\Leftrightarrow \quad cg^\circ = f^\circ d \qquad (\ gc^\circ = d^\circ f\ )$$

  $$\Rightarrow \quad g\langle R_c\rangle = R_d \qquad (\ gc^\circ cg^\circ = d^\circ d\ )$$

Regular pushouts

- $$\begin{array}{ccccc} R_c & \rightrightarrows & C & \overset{c}{\twoheadrightarrow} & A \\ \downarrow\uparrow & & g\downarrow\uparrow t & (1) & f\downarrow\uparrow s \\ R_d & \rightrightarrows & D & \underset{d}{\twoheadrightarrow} & B, \end{array}$$
  always a pushout

  $$g \cdot t = 1, \;\; f \cdot s = 1, \;\; c \cdot t = s \cdot d$$

- regular po: $\langle g, c \rangle : C \twoheadrightarrow D \times_B A$ regular epi

  $$\Leftrightarrow \quad cg^\circ = f^\circ d \qquad (\; gc^\circ = d^\circ f \;)$$

  $$\Rightarrow \quad g\langle R_c \rangle = R_d \qquad (\; gc^\circ cg^\circ = d^\circ d \;)$$

relative

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

Regular pushouts

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

- $$\begin{array}{ccccc} R_c & \rightrightarrows & C & \overset{c}{\twoheadrightarrow} & A \\ \downarrow\uparrow & & g \downarrow\uparrow t & (1) & f \downarrow\uparrow s \\ R_d & \rightrightarrows & D & \underset{d}{\twoheadrightarrow} & B, \end{array}$$

  always a pushout

  $$g \cdot t = 1, \;\; f \cdot s = 1, \;\; c \cdot t = s \cdot d$$

- regular po: $\langle g, c \rangle : C \twoheadrightarrow D \times_B A$ regular epi

  $$\Leftrightarrow \quad cg^\circ = f^\circ d \qquad (\; gc^\circ = d^\circ f \;)$$

  $$\Rightarrow \quad g\langle R_c \rangle = R_d \qquad (\; gc^\circ cg^\circ = d^\circ d \;)$$

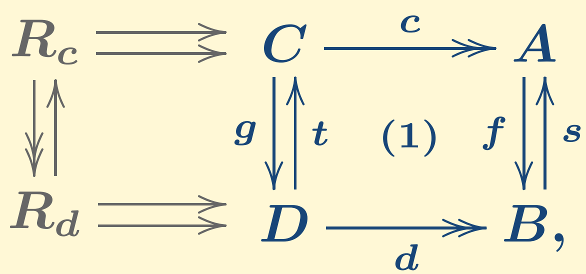
  relative

- Prop. [Bourn, *The denormalised* 3×3 *L*] $\mathcal{C}$ regular. TFAE:

  (a) $\mathcal{C}$ Mal'tsev cat

  (b) (1) always regular po

Regular pushouts

- 

  always a pushout

  $$g \cdot t = 1, \;\; f \cdot s = 1, \;\; c \cdot t = s \cdot d$$

- regular po: $\langle g, c\rangle : C \twoheadrightarrow D \times_B A$ regular epi

  $$\Leftrightarrow \quad cg^\circ = f^\circ d \qquad (\; gc^\circ = d^\circ f \;)$$

  $$\Rightarrow \quad g\langle R_c\rangle = R_d \qquad (\; gc^\circ cg^\circ = d^\circ d \;)$$

  relative

- Prop. [Bourn, *The denormalised* 3×3 *L*] $\mathcal{C}$ regular. TFAE:

  (a) $\mathcal{C}$ Mal'tsev cat

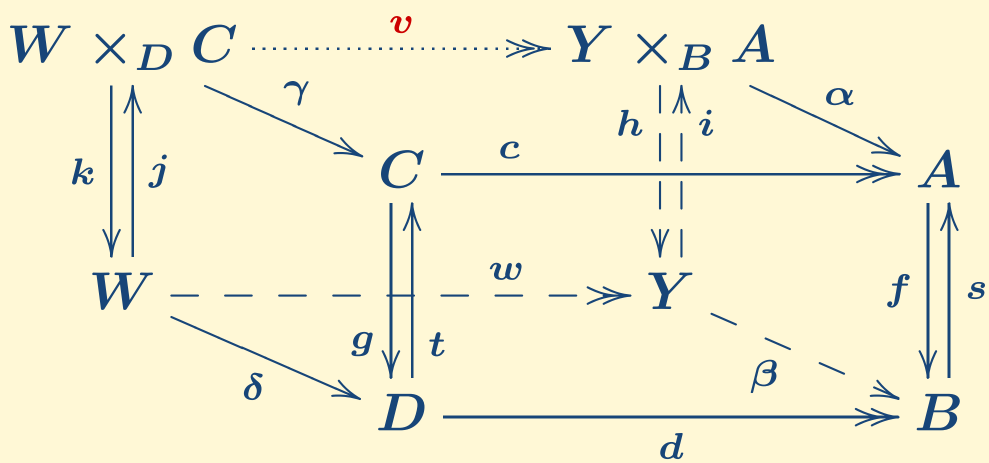
  (b) (1) always regular po

  Proof: calculus of relations

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

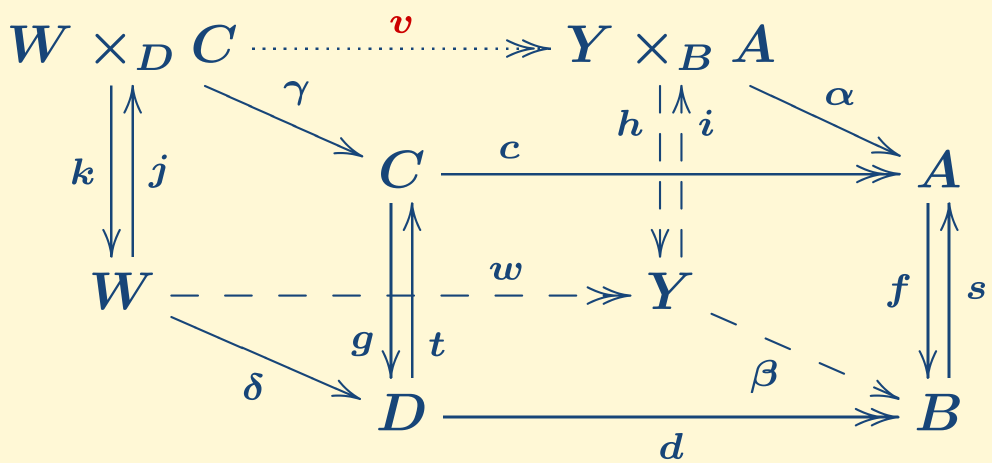
Stability property

- Lem. [Bourn, *The denormalised* $\mathbf{3} \times \mathbf{3}$ *L*] $\mathcal{C}$ regular Mal'tsev. In:



v is a regular epi

- Lem. [Bourn, *The denormalised* $\mathbf{3 \times 3}$ *L*] $\mathcal{C}$ regular Mal'tsev. In:

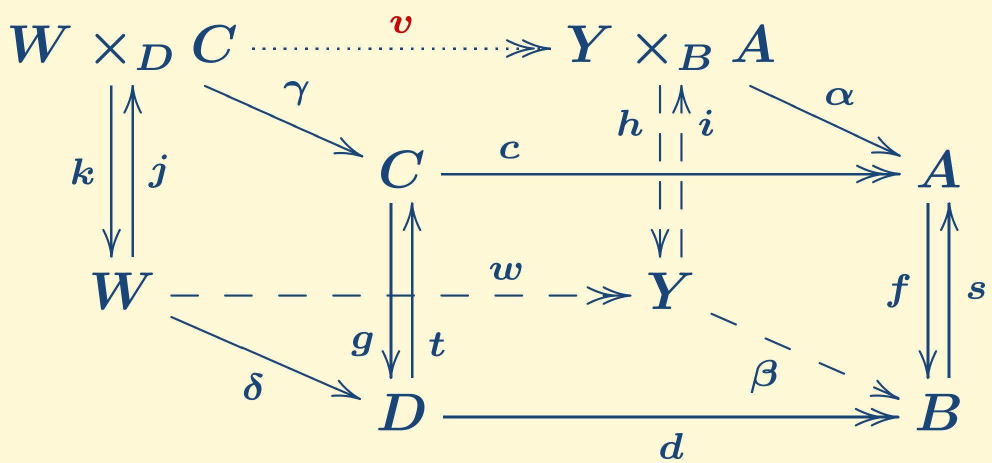


v is a regular epi

- Rem. $\alpha, \beta, \gamma, \delta$ arbitrary maps and c, d, w, v regular epis

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

- Lem. [Bourn, *The denormalised* 3×3 *L*] $\mathcal{C}$ regular Mal'tsev. In:



v is a regular epi

- Rem. $\alpha, \beta, \gamma, \delta$ arbitrary maps and c, d, w, v regular epis

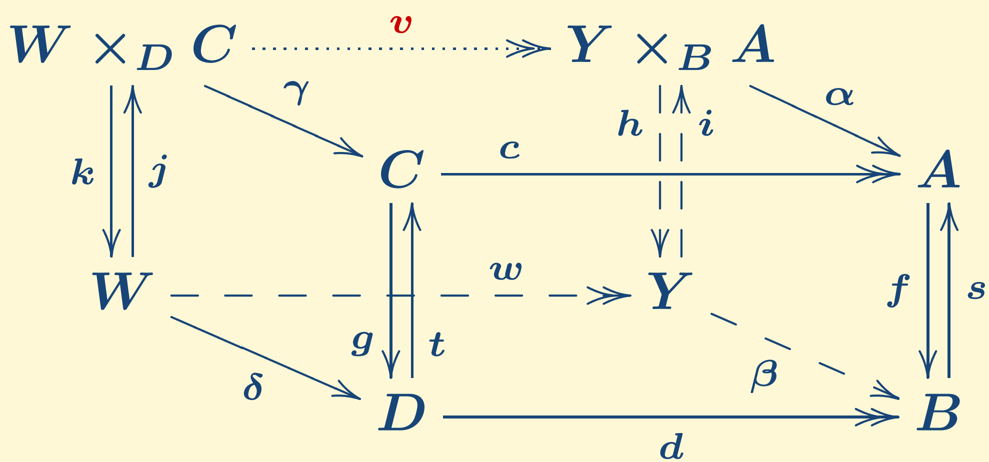
- Prop. $\mathcal{C}$ regular. TFAE:

  (a) $\mathcal{C}$ Mal'tsev cat

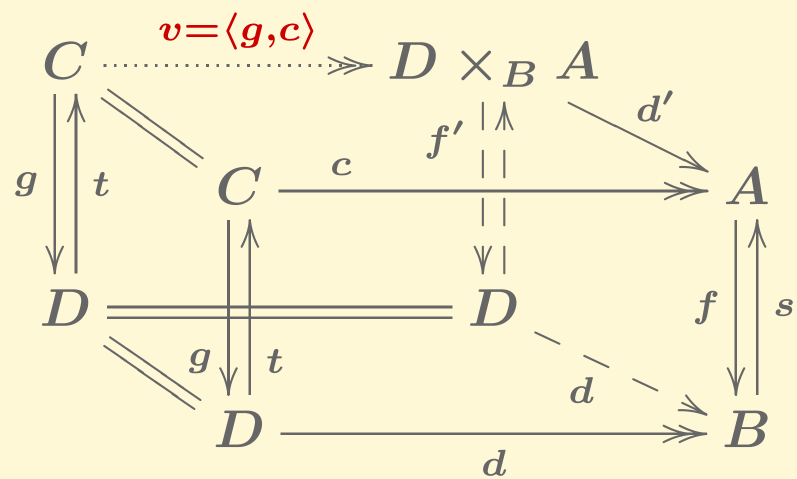
  (b) v is a regular epi

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

- **Lem.** [Bourn, *The denormalised* 3×3 *L*] $\mathcal{C}$ regular Mal'tsev. In:



v is a regular epi



- **Prop.** $\mathcal{C}$ regular. TFAE:
  - (a) $\mathcal{C}$ Mal'tsev cat
  - (b) v is a regular epi

Contents
Preliminaries
Regular Mal'tsev categories
Definition
Regular pushouts
Stability property
The Cuboid Lemma
The relative context

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3} \times \mathbf{3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

The Cuboid Lemma

- Aim. Mal'tsev $\Leftrightarrow$ stability pp $\Leftrightarrow$ Cuboid Lemma

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3} \times \mathbf{3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

- Aim. Mal'tsev $\Leftrightarrow$ stability pp $\Leftrightarrow$ Cuboid Lemma

- Regular

  Goursat cats Mal'tsev cats

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

Aim

- Aim. Mal'tsev $\Leftrightarrow$ stability pp $\Leftrightarrow$ Cuboid Lemma

- Regular

  Goursat cats

  ($\mathbf{3}$-permutable: $\boldsymbol{RSR = SRS}$)

  Mal'tsev cats

  ($\mathbf{2}$-permutable: $\boldsymbol{RS = SR}$)

Contents

Preliminaries

Regular Mal'tsev categories

The Cuboid Lemma

Aim

$\mathbf{3} \times \mathbf{3}$ Lemma

Stability properties

Cuboid Lemma

Main results

The relative context

- Aim. Mal'tsev $\Leftrightarrow$ stability pp $\Leftrightarrow$ Cuboid Lemma

- Regular

Goursat cats	Mal'tsev cats
($\mathbf{3}$-permutable: $RSR = SRS$)	($\mathbf{2}$-permutable: $RS = SR$)
	$\Leftrightarrow$ $\mathbf{(1)}$ regular po (Bourn)

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3} \times \mathbf{3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

- Aim. Mal'tsev $\Leftrightarrow$ stability pp $\Leftrightarrow$ Cuboid Lemma

- Regular

Goursat cats (**3**-permutable: $RSR = SRS$)	Mal'tsev cats (**2**-permutable: $RS = SR$)
	$\Leftrightarrow$ **(1)** regular po (Bourn) $\Rightarrow$ $\mathbf{3 \times 3}$ Lemma (Bourn)

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3} \times \mathbf{3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

- Aim. Mal'tsev $\Leftrightarrow$ stability pp $\Leftrightarrow$ Cuboid Lemma

- Regular

Goursat cats	Mal'tsev cats
(**3**-permutable: $RSR = SRS$)	(**2**-permutable: $RS = SR$)
	$\Leftrightarrow$ (1) regular po (Bourn)
$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Lack)	$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Bourn)

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3} \times \mathbf{3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

- Aim. Mal'tsev $\Leftrightarrow$ stability pp $\Leftrightarrow$ Cuboid Lemma

- Regular

Goursat cats (**3**-permutable: $RSR = SRS$)	Mal'tsev cats (**2**-permutable: $RS = SR$)
$\Leftrightarrow$ (1) Goursat po (GR) $\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Lack)	$\Leftrightarrow$ (1) regular po (Bourn) $\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Bourn)

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3} \times \mathbf{3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

Aim

- Aim. Mal'tsev $\Leftrightarrow$ stability pp $\Leftrightarrow$ Cuboid Lemma

- Regular

Goursat cats (**3**-permutable: $RSR = SRS$)	Mal'tsev cats (**2**-permutable: $RS = SR$)
$\Leftrightarrow$ (1) Goursat po (GR)	$\Leftrightarrow$ (1) regular po (Bourn)
$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Lack)	$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Bourn)
$\Leftrightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (GR)	

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3} \times \mathbf{3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

- Aim. Mal'tsev $\Leftrightarrow$ stability pp $\Leftrightarrow$ Cuboid Lemma

- Regular

Goursat cats ($\mathbf{3}$-permutable: $RSR = SRS$)	Mal'tsev cats ($\mathbf{2}$-permutable: $RS = SR$)
$\Leftrightarrow$ (1) Goursat po (GR)	$\Leftrightarrow$ (1) regular po (Bourn)
$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Lack)	$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Bourn)
$\Leftrightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (GR)	$\Leftrightarrow$?

Contents

Preliminaries

Regular Mal'tsev categories

The Cuboid Lemma
- Aim
- $\mathbf{3} \times \mathbf{3}$ Lemma
- Stability properties
- Cuboid Lemma
- Main results

The relative context

- Aim. Mal'tsev $\Leftrightarrow$ stability pp $\Leftrightarrow$ Cuboid Lemma

- Regular

Goursat cats (**3**-permutable: $RSR = SRS$)	Mal'tsev cats (**2**-permutable: $RS = SR$)
$\Leftrightarrow$ (1) Goursat po (GR)	$\Leftrightarrow$ (1) regular po (Bourn)
$\Rightarrow$ 3×3 Lemma (Lack)	$\Rightarrow$ 3×3 Lemma (Bourn)
$\Leftrightarrow$ 3×3 Lemma (GR)	$\Leftrightarrow$ Cuboid Lemma

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

- (denormalised) $\mathbf{3 \times 3}$ Lemma:

$$
\begin{array}{ccccc}
R_{\bar{g}} & \overset{t_1}{\underset{t_2}{\rightrightarrows}} & R_g & \xrightarrow{v} & R_f \\
{\scriptstyle \bar{g}_1}\downarrow\downarrow{\scriptstyle \bar{g}_2} & & {\scriptstyle g_1}\downarrow\downarrow{\scriptstyle g_2} & & {\scriptstyle f_1}\downarrow\downarrow{\scriptstyle f_2} \\
R_c & \overset{c_1}{\underset{c_2}{\rightrightarrows}} & A & \overset{c}{\twoheadrightarrow} & C \\
{\scriptstyle \bar{g}}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\
S & \overset{s_1}{\underset{s_2}{\rightrightarrows}} & B & \xrightarrow[d]{} & D
\end{array}
$$

upper row exact $\Leftrightarrow$ lower row exact

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

$\mathbf{3 \times 3}$ Lemma

- (denormalised) $\mathbf{3 \times 3}$ Lemma:

$$
\begin{array}{ccccc}
R_{\bar{g}} & \overset{t_1}{\underset{t_2}{\rightrightarrows}} & R_g & \xrightarrow{v} & R_f \\
{\scriptstyle \bar{g}_1} \downarrow\downarrow {\scriptstyle \bar{g}_2} & & {\scriptstyle g_1} \downarrow\downarrow {\scriptstyle g_2} & & {\scriptstyle f_1} \downarrow\downarrow {\scriptstyle f_2} \\
R_c & \overset{c_1}{\underset{c_2}{\rightrightarrows}} & A & \overset{c}{\twoheadrightarrow} & C \\
{\scriptstyle \bar{g}} \downarrow & & {\scriptstyle g} \downarrow & & {\scriptstyle f} \downarrow \\
S & \overset{s_1}{\underset{s_2}{\rightrightarrows}} & B & \xrightarrow[d]{} & D
\end{array}
$$

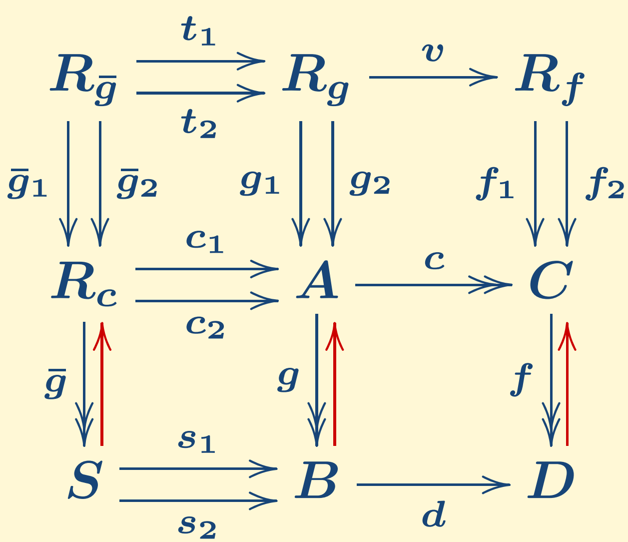
upper row exact $\Leftrightarrow$ lower row exact

- Gran, Rodelo: Goursat $\Leftrightarrow$ $\mathbf{3 \times 3}$ Lemma

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3 \times 3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

$\mathbf{3} \times \mathbf{3}$ Lemma

- (denormalised) $\mathbf{3} \times \mathbf{3}$ Lemma:



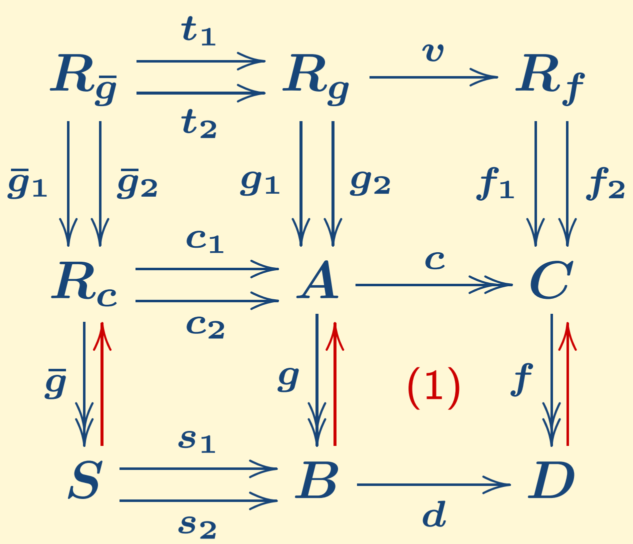
upper row exact $\Leftrightarrow$ lower row exact

- Gran, Rodelo: Goursat $\Leftrightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma $\Leftrightarrow$ split $\mathbf{3} \times \mathbf{3}$ Lemma

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3} \times \mathbf{3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

3×3 Lemma

- (denormalised) $\mathbf{3} \times \mathbf{3}$ Lemma:



upper row exact $\Leftrightarrow$ lower row exact

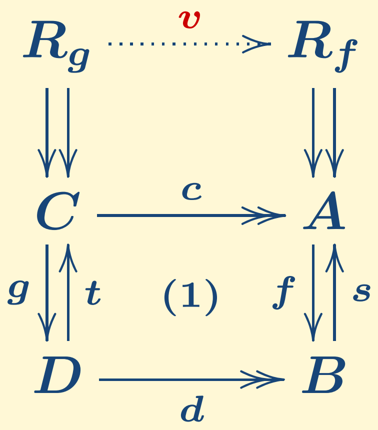
- Gran, Rodelo: Goursat $\Leftrightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma $\Leftrightarrow$ split $\mathbf{3} \times \mathbf{3}$ Lemma

$\uparrow$

Goursat po

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3} \times \mathbf{3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

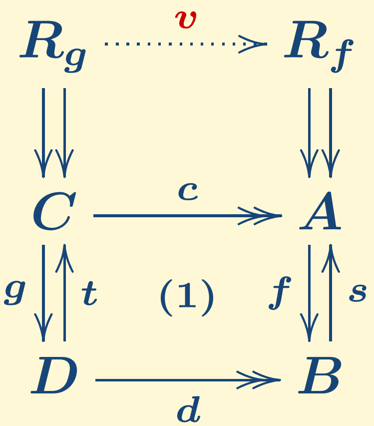
- Goursat po



v is a regular epi

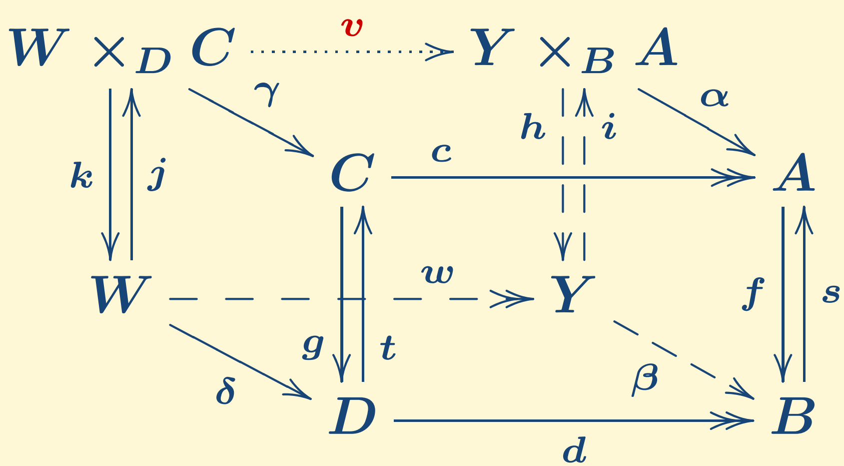
Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

- Goursat po

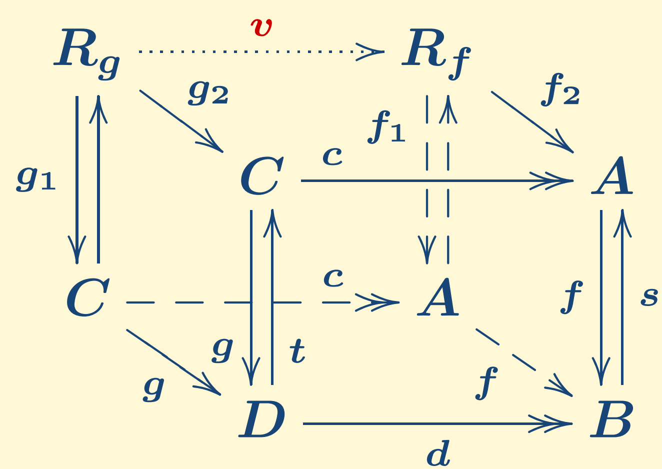


v is a regular epi

- Mal'tsev vs Goursat



split pbs



kernel pairs

Contents

Preliminaries

Regular Mal'tsev categories

The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results

The relative context

Cuboid Lemma

Contents

Preliminaries

Regular Mal'tsev categories

The Cuboid Lemma

Aim

3×3 Lemma

Stability properties

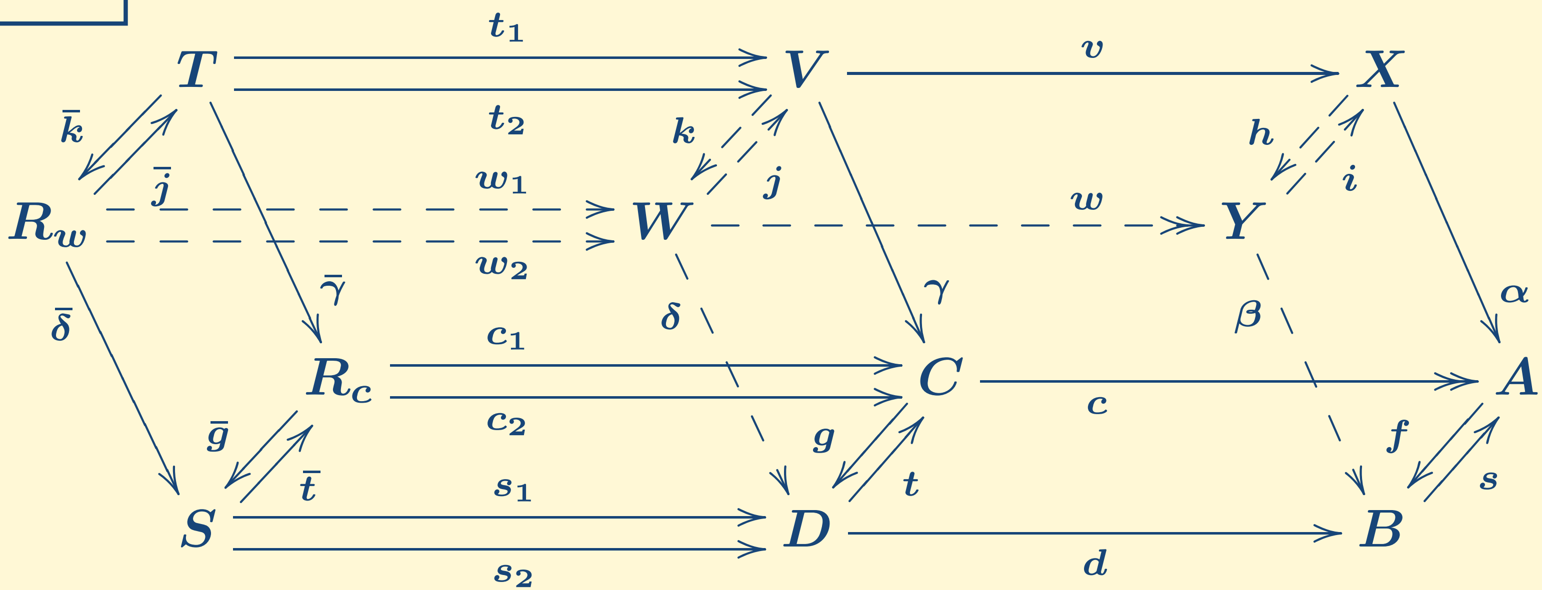
Cuboid Lemma

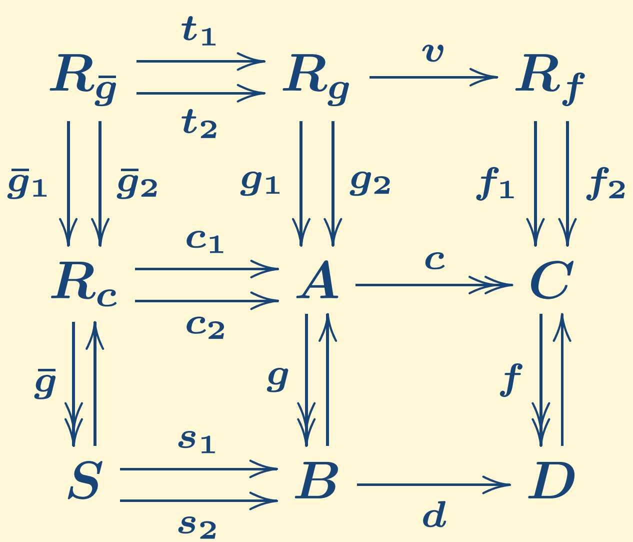
Main results

The relative context

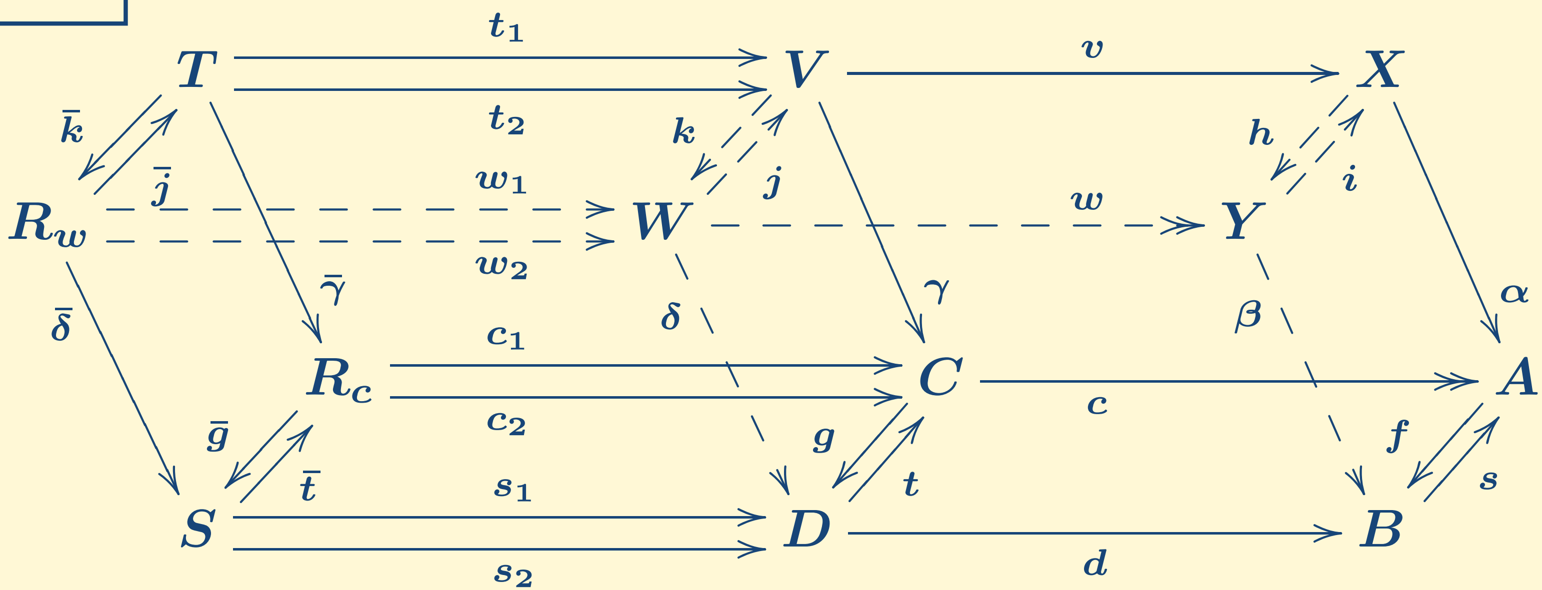
- $$\begin{array}{ccccc}
R_{\bar{g}} & \overset{t_1}{\underset{t_2}{\rightrightarrows}} & R_g & \xrightarrow{v} & R_f \\
{\scriptstyle \bar{g}_1}\downdownarrows{\scriptstyle \bar{g}_2} & & {\scriptstyle g_1}\downdownarrows{\scriptstyle g_2} & & {\scriptstyle f_1}\downdownarrows{\scriptstyle f_2} \\
R_c & \overset{c_1}{\underset{c_2}{\rightrightarrows}} & A & \overset{c}{\twoheadrightarrow} & C \\
{\scriptstyle \bar{g}}\downarrow\uparrow & & {\scriptstyle g}\downarrow\uparrow & & {\scriptstyle f}\downarrow\uparrow \\
S & \overset{s_1}{\underset{s_2}{\rightrightarrows}} & B & \xrightarrow[d]{} & D
\end{array}$$

Cuboid Lemma

- 

- 

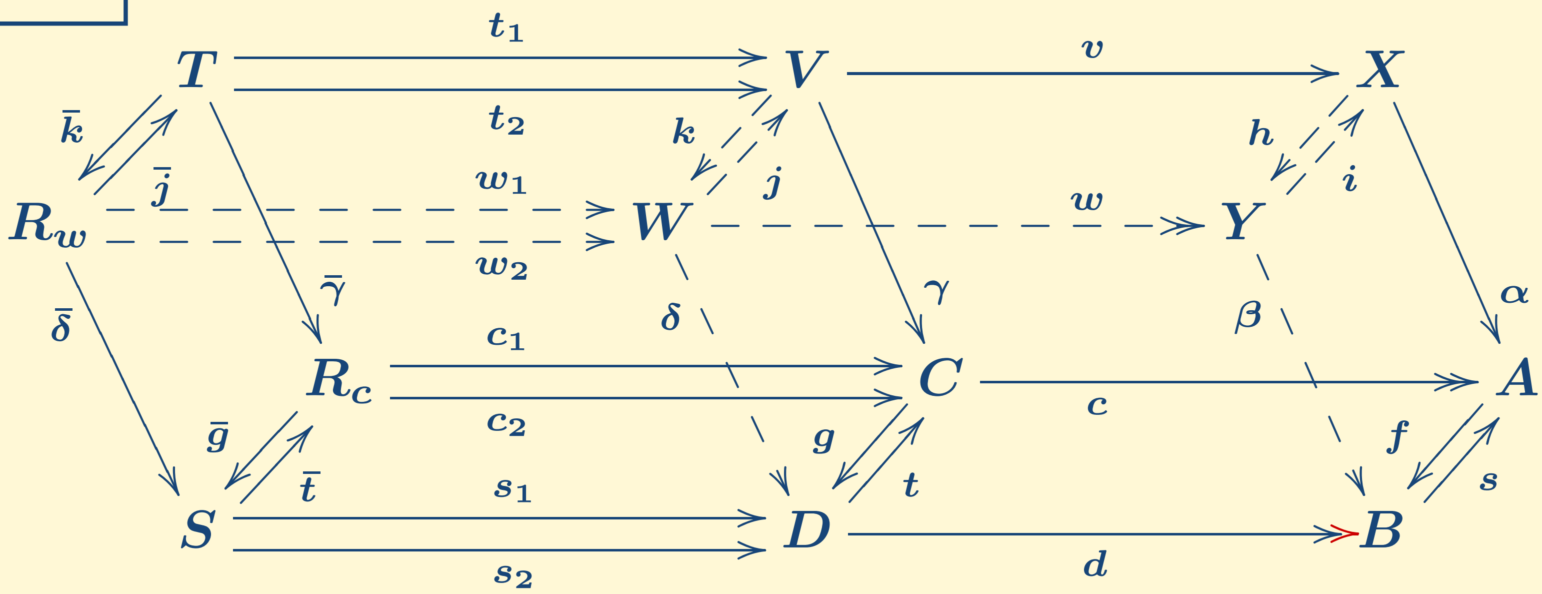
Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

- 

upper row exact $\Leftrightarrow$ lower row exact

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

Cuboid Lemma



upper row exact $\Leftrightarrow$ lower row exact

- Rem. d regular epi, $d \cdot s_1 = d \cdot s_2$, $v \cdot t_1 = v \cdot t_2$

Contents

Preliminaries

Regular Mal'tsev categories

The Cuboid Lemma

Aim

3×3 Lemma

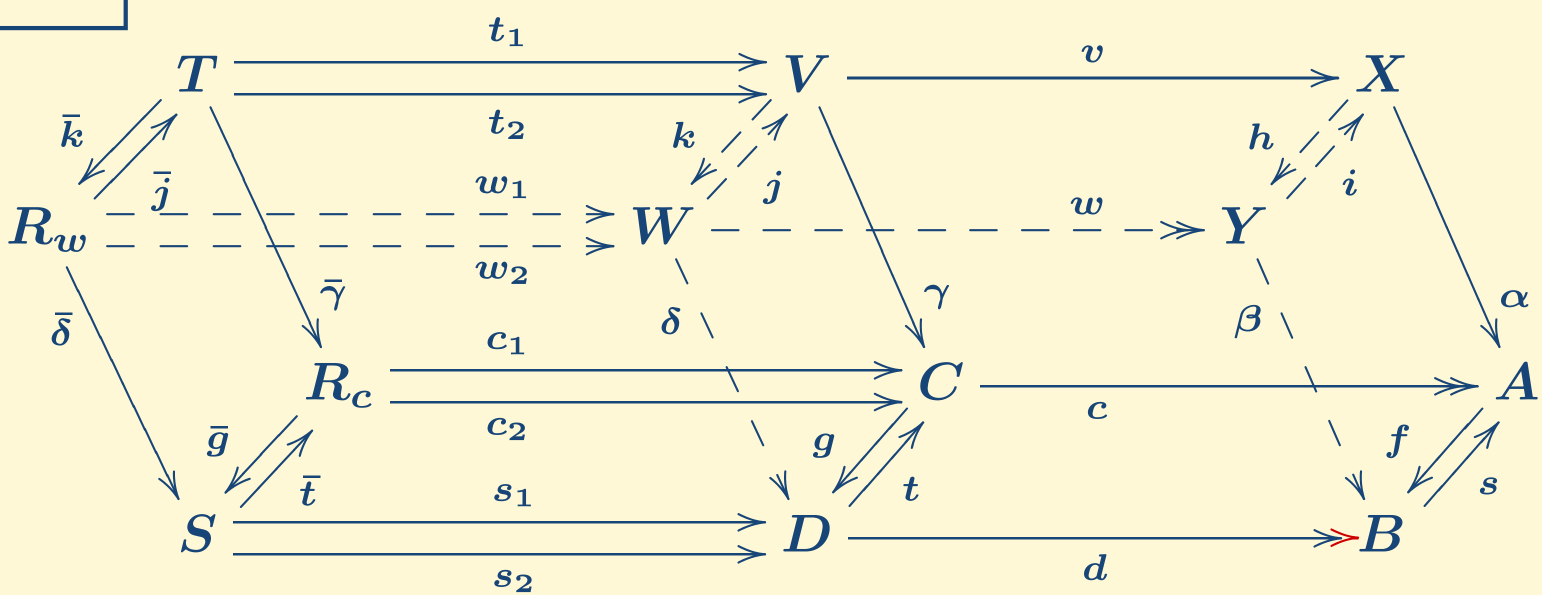
Stability properties

Cuboid Lemma

Main results

The relative context

Cuboid Lemma

- 

upper row exact $\Leftrightarrow$ lower row exact

- Rem. d regular epi, $d \cdot s_1 = d \cdot s_2$, $v \cdot t_1 = v \cdot t_2$

- lower row exact $\Rightarrow$ upper row exact: v is a regular epi

  upper row exact $\Rightarrow$ lower row exact: $S = R_d$

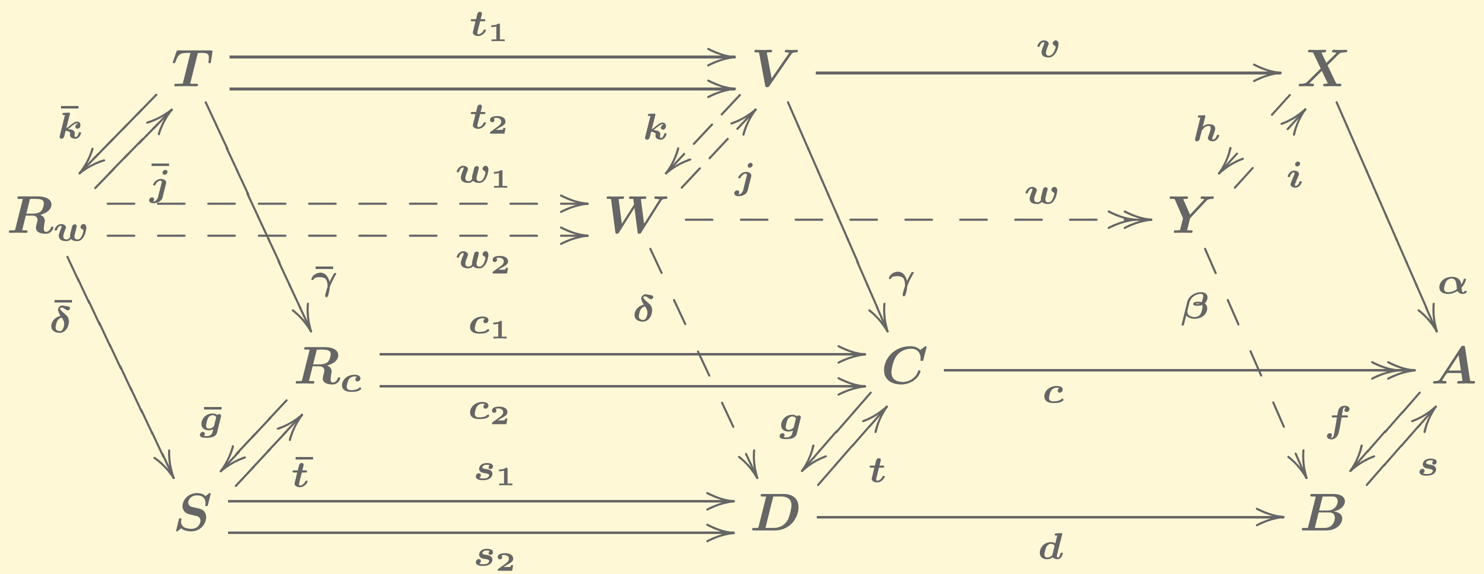
Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

Main results

- **Thm.** $\mathcal{C}$ regular. TFAE:
  - (a) $\mathcal{C}$ Mal'tsev cat
  - (b) Cuboid Lemma

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
$\mathbf{3} \times \mathbf{3}$ Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

- Thm. $\mathcal{C}$ regular. TFAE:
  - (a) $\mathcal{C}$ Mal'tsev cat
  - (b) Cuboid Lemma



(a) ⇒ (b) · lower row exact ⇒ upper row exact ?

- stability pp: c, d, w regular epis ⇒ v regular epi

Contents

Preliminaries

Regular Mal'tsev categories

The Cuboid Lemma

Aim

3×3 Lemma

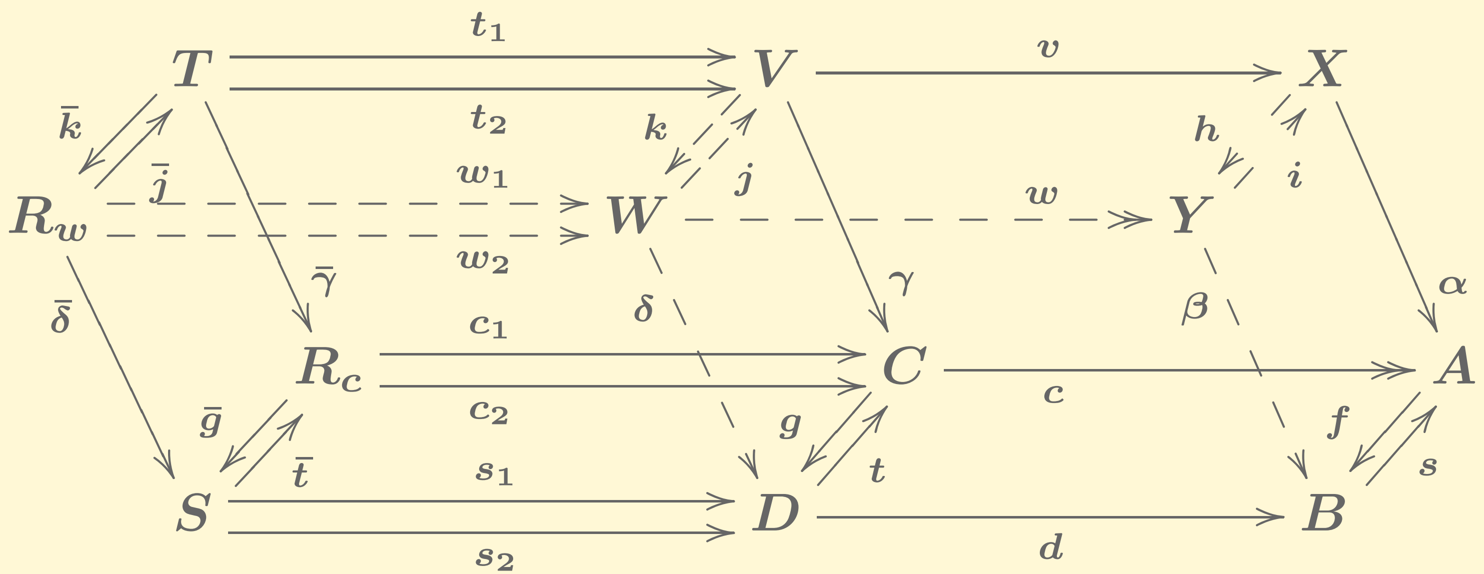
Stability properties

Cuboid Lemma

Main results

The relative context

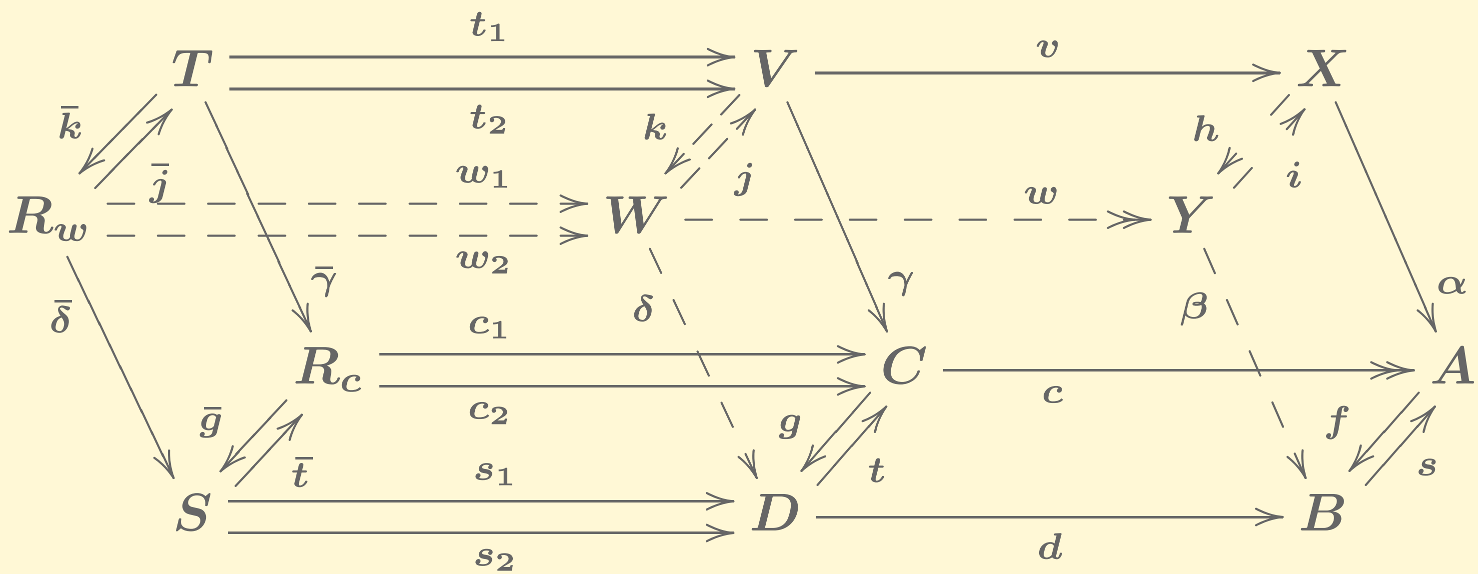
- Thm. $\mathcal{C}$ regular. TFAE:
  - (a) $\mathcal{C}$ Mal'tsev cat
  - (b) Cuboid Lemma



(a) $\Rightarrow$ (b)
- lower row exact $\Rightarrow$ upper row exact ?
  - stability pp: c, d, w regular epis $\Rightarrow$ v regular epi
- upper row exact $\Rightarrow$ lower row exact ?
  - **always true!**

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

- Thm. $\mathcal{C}$ regular. TFAE:
  - (a) $\mathcal{C}$ Mal'tsev cat
  - (b) Cuboid Lemma



(b) ⇒ (a) · cube with stability pp ?

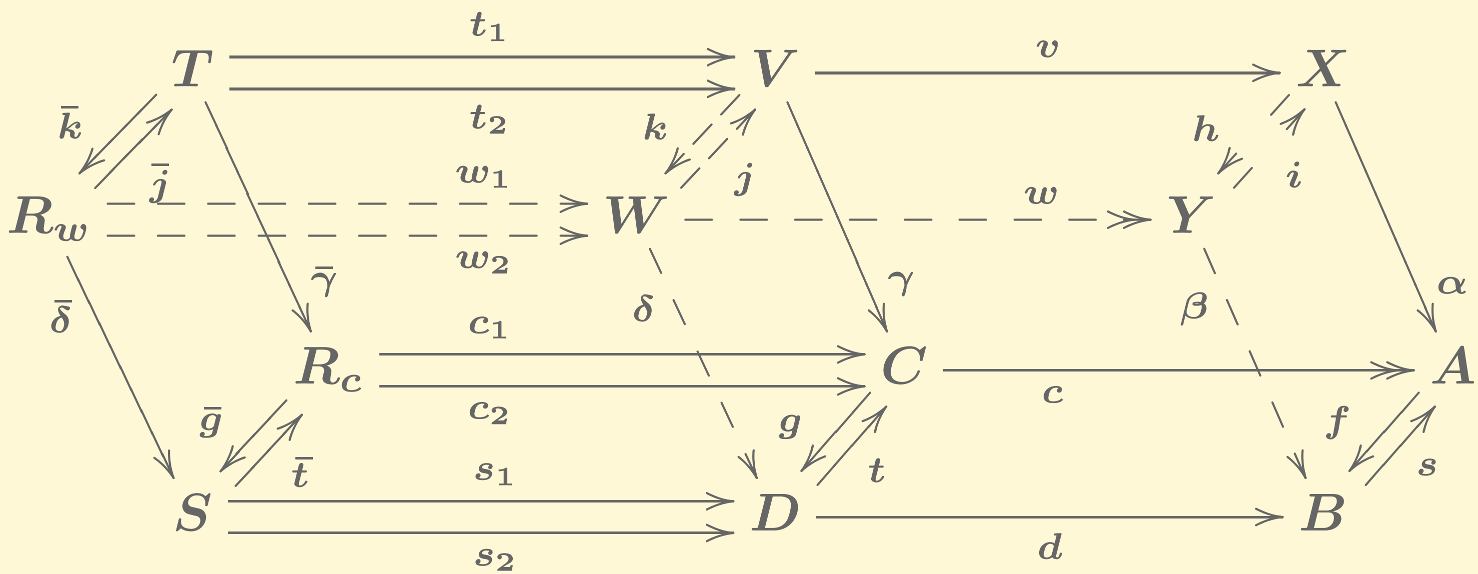
- $R_w, R_c, S = R_d,\ T = R_w \times_{R_d} R_c$
- lower row exact ⇒ upper row exact

} v regular epi

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

Main results

- Thm. $\mathcal{C}$ regular. TFAE:
  - (a) $\mathcal{C}$ Mal'tsev cat
  - (b) Cuboid Lemma



(b) ⇒ (a) · cube with stability pp ?

- $R_w, R_c, S = R_d, T = R_w \times_{R_d} R_c$
- lower row exact ⇒ upper row exact

} v regular epi

Upper Cuboid Lemma

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

Main results

- Thm. $\mathcal{C}$ regular. TFAE:
  - (a) $\mathcal{C}$ Mal'tsev cat
  - (b) Cuboid Lemma
  - (c) Upper Cuboid Lemma

(b) $\Rightarrow$ (a) · cube with stability pp ?

- $R_w, R_c, S = R_d$, $T = R_w \times_{R_d} R_c$
- lower row exact $\Rightarrow$ upper row exact

$\Big\}$ v regular epi

Upper Cuboid Lemma

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
Aim
3×3 Lemma
Stability properties
Cuboid Lemma
Main results
The relative context

Contents

Preliminaries

Regular Mal'tsev categories

The Cuboid Lemma

The relative context
From the absolute to the relative context

The relative context

From the absolute to the relative context

- absolute context $\rightsquigarrow$ relative context

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
The relative context
From the absolute to the relative context

From the absolute to the relative context

- absolute context $\rightsquigarrow$ relative context

- Regular

Goursat cats	Mal'tsev cats
($\mathbf{3}$-permutable: $RSR = SRS$)	($\mathbf{2}$-permutable: $RS = SR$)
$\Leftrightarrow$ $\mathbf{(1)}$ Goursat po (GR)	$\Leftrightarrow$ $\mathbf{(1)}$ regular po (Bourn)
$\Rightarrow$ $\mathbf{3 \times 3}$ Lemma (Lack)	$\Rightarrow$ $\mathbf{3 \times 3}$ Lemma (Bourn)
$\Leftrightarrow$ $\mathbf{3 \times 3}$ Lemma (GR)	$\Leftrightarrow$ Cuboid Lemma (GR)

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
The relative context
From the absolute to the relative context

From the absolute to the relative context

- absolute context $\rightsquigarrow$ relative context

  lex + $\mathcal{E}$ class of regular epis sth ...
- relative Regular (T. Janelidze)

Goursat cats (**3**-permutable: $RSR = SRS$)	Mal'tsev cats (**2**-permutable: $RS = SR$)
$\Leftrightarrow$ (1) Goursat po (GR)	$\Leftrightarrow$ (1) regular po (Bourn)
$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Lack)	$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Bourn)
$\Leftrightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (GR)	$\Leftrightarrow$ Cuboid Lemma (GR)

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
The relative context
From the absolute to the relative context

From the absolute to the relative context

- absolute context $\rightsquigarrow$ relative context

lex + $\mathcal{E}$ class of regular epis sth ...

- relative Regular (T. Janelidze)

Goursat cats (**3**-permutable: $RSR = SRS$)	Mal'tsev cats (**2**-permutable: $RS = SR$)
$\Leftrightarrow$ (1) Goursat po (GR)	$\Leftrightarrow$ (1) regular po (Bourn)
$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Lack)	$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Bourn)
$\Leftrightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (GR)	$\Leftrightarrow$ Cuboid Lemma (GR)

relative version

(Goedecke, T. Janelidze)

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
The relative context
From the absolute to the relative context

From the absolute to the relative context

- absolute context $\rightsquigarrow$ relative context

lex + $\mathcal{E}$ class of regular epis sth ...

- relative Regular (T. Janelidze)

relative Goursat cats (**3**-permutable: $RSR = SRS$) $\mathcal{E}$-relations	Mal'tsev cats (**2**-permutable: $RS = SR$)
$\Leftrightarrow$ **(1)** Goursat po (GR)	$\Leftrightarrow$ **(1)** regular po (Bourn)
$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Lack)	$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Bourn)
$\Leftrightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (GR)	$\Leftrightarrow$ Cuboid Lemma (GR)

relative version

(Goedecke, T. Janelidze)

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
The relative context
From the absolute to the relative context

From the absolute to the relative context

- absolute context $\rightsquigarrow$ relative context

lex + $\mathcal{E}$ class of regular epis sth ...

- relative Regular (T. Janelidze)

relative Goursat cats	Mal'tsev cats
($\mathbf{3}$-permutable: $RSR = SRS$) $\mathcal{E}$-relations	($\mathbf{2}$-permutable: $RS = SR$)
$\Leftrightarrow$ $(\mathbf{1})$ Goursat po (GR)	$\Leftrightarrow$ $(\mathbf{1})$ regular po (Bourn)
$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Lack)	$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Bourn)
$\Leftrightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (GR)	$\Leftrightarrow$ Cuboid Lemma (GR)

relative version

(Goedecke, T. Janelidze)

calculus of relations $\rightsquigarrow$ calculus of $\mathcal{E}$-relations

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
The relative context
From the absolute to the relative context

From the absolute to the relative context

- absolute context $\rightsquigarrow$ relative context

lex + $\mathcal{E}$ class of regular epis sth ...

- relative Regular (T. Janelidze)

relative Goursat cats	relative Mal'tsev cats
($\mathbf{3}$-permutable: $\underbrace{RSR = SRS}_{\mathcal{E}\text{-relations}}$)	($\mathbf{2}$-permutable: $\underbrace{RS = SR}_{\mathcal{E}\text{-relations}}$)
$\Leftrightarrow$ $\mathbf{(1)}$ Goursat po (GR)	$\Leftrightarrow$ $\mathbf{(1)}$ regular po (Bourn)
$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Lack)	$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Bourn)
$\Leftrightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (GR)	$\Leftrightarrow$ Cuboid Lemma (GR)

relative version

(Goedecke, T. Janelidze)

calculus of relations $\rightsquigarrow$ calculus of $\mathcal{E}$-relations

Contents

Preliminaries

Regular Mal'tsev categories

The Cuboid Lemma

The relative context

From the absolute to the relative context

From the absolute to the relative context

Contents
Preliminaries
Regular Mal'tsev categories
The Cuboid Lemma
The relative context
From the absolute to the relative context

- absolute context $\rightsquigarrow$ relative context

lex + $\mathcal{E}$ class of regular epis sth ...

- relative Regular (T. Janelidze)

relative Goursat cats	relative Mal'tsev cats
($\mathbf{3}$-permutable: $\boldsymbol{RSR = SRS}$) $\mathcal{E}$-relations	($\mathbf{2}$-permutable: $\boldsymbol{RS = SR}$) $\mathcal{E}$-relations
$\Leftrightarrow$ $\mathbf{(1)}$ Goursat po (GR)	$\Leftrightarrow$ $\mathbf{(1)}$ regular po (Bourn)
$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Lack)	$\Rightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (Bourn)
$\Leftrightarrow$ $\mathbf{3} \times \mathbf{3}$ Lemma (GR)	$\Leftrightarrow$ Cuboid Lemma (GR)
relative version (Goedecke, T. Janelidze)	relative version (Everaert, Goedecke T. Janelidze, Van der Linden)

calculus of relations $\rightsquigarrow$ calculus of $\mathcal{E}$-relations