Combinatorial Categories

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Definition 1. A *combinatorial category* is a locally presentable category $\mathcal{K}$ equipped with a set $\mathcal{X}$ of morphisms.

Cellular morphisms are transfinite compositions of pushouts of morphisms from $\mathcal{X}$.

Cofibrant morphisms are retracts of cellular ones in $\mathcal{K}^2$.

$$\mathrm{cof}(\mathcal{X}) = \mathrm{Rt}\,\mathrm{cell}(\mathcal{X})$$

κ-*combinatorial* means that $\mathcal{K}$ is locally κ-presentable and $\mathcal{X} \subseteq (K_\kappa)^2$.

A combinatorial category $\mathcal{K}$ is equipped with a weak factorization system

$$(\mathrm{cof}(\mathcal{X}), \mathcal{X}^{\square})$$

A weak factorization of the codiagonal

$$\nabla : K + K \xrightarrow{c_K} C(K) \xrightarrow{s_K} K$$

provides the cylinder object $C(K)$ for K.

Thus we can do homotopy theory in $\mathcal{K}$.

The adjective "combinatorial" has the same meaning as for model categories.

The term "cofibration category" is occupied by categories equipped with cofibrations and weak equivalences.

Proposition 1. (Lurie) Let $\mathcal{K}$ be a κ-combinatorial category. Then

$$\mathsf{cof}(\mathcal{X}) = \mathsf{cell}\,\mathsf{cof}_\kappa(\mathcal{X}).$$

Here, $\mathsf{cof}_\kappa(\mathcal{X}) = \mathsf{cof}(\mathcal{X}) \cap (\mathcal{K}_\kappa)^2$.

The result means that Rt and cell can be interchanged.

The proof is quite complex and uses good colimits.

A poset P is good if it is well-founded and has a least element $\perp$.

Well-ordeder sets and shape posets for pushouts are good.

An element $x \in P$ is *isolated* if there is a top element x^- strictly below x.

A non-isolated element distinct from $\perp$ is called *limit*.

A *good* diagram $D : P \to \mathcal{K}$ is such that Dx is a colimit of the restriction of D on elements strictly below x for each limit x.

The *composition* of D is the component $\delta_\perp$ of the colimit cocone.

Links of D are morphisms $D(x^- \to x)$ for x isolated.

Proposition 2. (Lurie) Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$. Then the composition of a good diagram with links in $\mathcal{L}$ belongs to $\mathcal{L}$.

There is a stronger result; $\mathrm{Po}(\mathcal{X})$ denotes pushouts of morphisms from $\mathcal{X}$.

Proposition 3. Let $\mathcal{X}$ be a class of morphisms in a cocomplete category $\mathcal{K}$. Then the composition of a good diagram with links in $\mathrm{Po}(\mathcal{X})$ belongs to $\mathrm{cell}(\mathcal{X})$.

A good poset is κ-*good* if all its principal ideals $\downarrow x$ have cardinality $< \kappa$.

Proposition 4. Let $(\mathcal{K}, \mathcal{X})$ be a κ-combinatorial category. Then every cellular morphism is a composition of a κ-good κ-directed diagram with links in $\mathsf{Po}(\mathcal{X})$.

This result may be called a *fat small object argument* because it replaces a thin transfinite composition containing large objects by a fat good composition of small objects.

Let $\mathsf{Po}_\kappa(\mathcal{X}) = \mathsf{Po}(\mathcal{X}) \cap (\mathcal{K}_\kappa)^2$.

Corollary 1. Let $(\mathcal{K}, \mathcal{X})$ be a κ-combinatorial category. Then every cellular morphism with the domain in $\mathcal{K}_\kappa$ is a composition of a κ-good κ-directed diagram with links in $\mathsf{Po}_\kappa(\mathcal{X})$.

An object K of a combinatorial category is *cofibrant* if a unique morphism $0 \to K$ from an initial object is cofibrant.

Corollary 2. Any cofibrant object in a κ-combinatorial category is a κ-filtered colimit of κ-presentable cofibrant objects.

A functor F between combinatorial categories is called *combinatorial* if it preserves colimits and cofibrant morphisms.

Combinatorial functors are left adjoints and correspond to left Quillen functors between model categories.

Theorem. COMB is closed in CAT under PIE-limits.

This extends the Limit Theorem of M. Makkai and R. Paré. Good colimits are an indispensable tool.

Consequently, COMB is closed under pseudolimits and lax limits.

Corollary 3. (Lurie) Let $\mathcal{K}$ be a combinatorial category and $\mathcal{C}$ a small category. Then $\mathcal{K}^{\mathcal{C}}$ is combinatorial (with respect to pointwise cofibrant morphisms).

Let $\mathcal{K}$ be the category of left R-modules over a ring R. A monomorphism f is called an $\mathcal{S}$-monomorphism if its cokernel belongs to $\mathcal{S} \subseteq \mathcal{K}$.

An object K is *$\mathcal{S}$-filtered* if a unique morphism $0 \to K$ is a transfinite composition of $\mathcal{S}$-monomorphisms.

For example, if $\mathcal{S}$ consists of simple modules then $\mathcal{S}$-filterd modules are precisely semiartinian ones, i.e., those belonging to the localizing subcategory generated by simple modules.

A class $\mathcal{C}$ is *deconstructible* if it is a class of $\mathcal{S}$-filtered modules for a set $\mathcal{S}$.

Proposition 5. A class $\mathcal{C}$ is deconstructible if and only if $\mathcal{K}$ is a combinatorial category with cellular morphisms being $\mathcal{C}$-monomorphisms.

Objects of $\mathcal{C}$ are precisely cellular objects.

This relates module theoretic investigations (Gillespie, Trlifaj, Šťovíček) to our framework. Good colimits are replaced there by the use of generalized Hill lemma.

Corollary 4. (Šťovíček) Let $\mathcal{C}$ be a deconstructible class in $\mathcal{K}$. Then $\mathrm{Comp}(\mathcal{C})$ is deconstructible in $\mathrm{Comp}(\mathcal{K})$.

Problem. Can the Theorem be extended to model categories?