Enough regular Cauchy filters for asymmetric uniform and pointfree structures

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The symmetric case of nearness frames:

Uniform $\Rightarrow$ strong $\Rightarrow$ has enough regular Cauchy filters.

- Completion is a coreflection for uniform frames (Isbell).
- Completion is a coreflection for strong nearness frames (Banaschewski, Hong, Pultr).
- Completion is *not* a coreflection for nearness frames (Banaschewski, Hong, Pultr).

Definition

A quasi-nearness biframe $(L, \mathcal{U}L)$ has enough regular Cauchy bifilters, or is erc if, whenever $\varphi : L \to T$ is a Cauchy bifilter, there exists a regular Cauchy bifilter $\psi : L \to T$ such that $\psi \leq \varphi$, meaning that $\psi(a) \leq \varphi(a)$ for all $a \in L_0$.

Definition

Let $\varphi : L \to T$ be a bifilter on $(L, \mathcal{U}L)$. We define its regular reduction, φ°, as follows. For $a \in L_0$,

$$\varphi^\circ(a) = \bigvee \{\varphi(s) \wedge \varphi(t) : (s,t) \lhd (x,y), x \wedge y \leq a\}.$$

Here $(s,t) \lhd (x,y)$ is taken to mean that $(s,t),(x,y) \in L_1 \times L_2$ and that $s \lhd_1 x$ and $t \lhd_2 y$.

Let $(L, \mathcal{U}L)$ be a quasi-nearness biframe.

Lemma

Let $\varphi, \psi : L \to T$ be Cauchy bifilters, with $\psi \leq \varphi$. Then $\varphi^\circ \leq \psi$.

Corollary

Let $\varphi, \psi : L \to T$ be bifilters, with $\psi \leq \varphi$ and ψ a regular Cauchy bifilter. Then $\psi = \varphi^\circ$.

Corollary

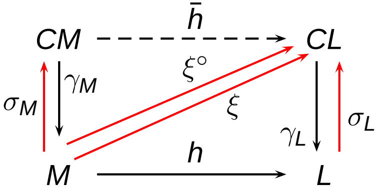
A quasi-nearness biframe $(L, \mathcal{U}L)$ has enough regular Cauchy bifilters if and only if, for any Cauchy bifilter $\varphi : L \to T$, $\varphi^\circ : L \to T$ is a regular Cauchy bifilter.

Questions of functoriality

Let $h : (M, \mathcal{U}M) \to (L, \mathcal{U}L)$ be uniform and $(M, \mathcal{U}M)$ have enough regular Cauchy bifilters. We seek a biframe map $\bar{h}$ making this diagram commute:

$$\begin{array}{ccc} CM & \overset{\bar{h}}{\dashrightarrow} & CL \\ {\scriptstyle \gamma_M}\downarrow & & \downarrow{\scriptstyle \gamma_L} \\ M & \xrightarrow{\quad h \quad} & L \end{array}$$

We use the right adjoints of the completion maps:



It remains to show that $\gamma_L \bar{h} = h\gamma_M$ by showing that

$$\gamma_L \bar{h}(\sigma_M(a)) = h\gamma_M(\sigma_M(a)) \text{ for } a \in M_0.$$

This suffices because $\{\sigma_M(a) : a \in M_0\}$ generates the total part of CM.

Definition

We call $(M, \mathcal{U}M)$ strong if $\check{C} \in \mathcal{U}M$ whenever $C \in \mathcal{U}M$, where $\check{C} = \{(d, \tilde{d}) : (d, \tilde{d}) \lhd (c, \tilde{c}) \text{ for some } (c, \tilde{c}) \in C\}$.

Let $h : (M, \mathcal{U}M) \to (L, \mathcal{U}L)$ be uniform and $(M, \mathcal{U}M)$ be strong and erc. Then the biframe map $\bar{h}$ is uniform:

$$\begin{array}{ccc}
(CM, C\mathcal{U}M) & \xrightarrow{\quad \bar{h} \quad} & (CL, C\mathcal{U}L) \\
\sigma_M \uparrow \; \downarrow \gamma_M & & \gamma_L \downarrow \; \uparrow \sigma_L \\
(M, \mathcal{U}M) & \xrightarrow{\quad h \quad} & (L, \mathcal{U}L)
\end{array}$$

One checks that, for $C \in \mathcal{U}M$, $\sigma_L h(\check{C}) \leq \bar{h}\sigma_M(C)$.

Lemma

- $(M, \mathcal{U}M)$ has enough regular Cauchy bifilters iff $C(M, \mathcal{U}M)$ has enough regular Cauchy bifilters.
- $(M, \mathcal{U}M)$ is strong iff $C(M, \mathcal{U}M)$ is strong.
- $(M, \mathcal{U}M)$ is quasi-uniform iff $C(M, \mathcal{U}M)$ is quasi-uniform.

Theorem

Denote by **ESQ** the subcategory of quasi-nearness biframes that have enough regular Cauchy bifilters and are strong. The quasi-complete objects of **ESQ** form a coreflective subcategory of **ESQ**.

Characterizations and consequences

Theorem

For any quasi-nearness biframe $(M, \mathcal{U}M)$, the following are equivalent:

(a) $(M, \mathcal{U}M)$ has enough regular Cauchy bifilters.

(b) $\delta^\circ : M \to CFM$ is a regular Cauchy bifilter on $(M, \mathcal{U}M)$, where δ is the the universal Cauchy bifilter.

(c) The quotient $\tau : CFM \to CM$ has a right inverse; that is, there exists a biframe map $f : CM \to CFM$ such that $\tau f = \mathrm{id}_{CM}$.

If $(M, \mathcal{U}M)$ is totally bounded (meaning that the finite uniform paircovers generate the quasi-nearness), then the nucleus defining the Cauchy filter quotient CFM is:

$$k(U) = \bigcup\{\downarrow(x \wedge y) : x \in M_1, y \in M_2, x \wedge y \wedge C^s \subseteq U \text{ for some } C \in \mathcal{U}M\}$$

where $x \wedge y \wedge C^s = \{x \wedge y \wedge c \wedge \tilde{c} : (c, \tilde{c}) \in C\}$.

Theorem

If $(M, \mathcal{U}M)$ is totally bounded, then CFM is compact.

Corollary

If $(M, \mathcal{U}M)$ is totally bounded and has enough regular Cauchy bifilters, then $(M, \mathcal{U}M)$ is quasi-uniform and $C(M, \mathcal{U}M)$ is compact.

Example

This is an example of a totally bounded quasi-uniform biframe that does not have enough regular Cauchy bifilters.

Let $L = (\mathcal{D}\mathbb{R}, \mathcal{O}\mathbb{R}, \mathcal{D}\mathbb{R})$ where $\mathcal{D}\mathbb{R}$ is the discrete topology on the reals and $\mathcal{O}\mathbb{R}$ is the usual topology on the reals. Let $\mathcal{U}L$ be the standard binary quasi-nearness on L, generated by $\{C^{UV} : U \cup V = \mathbb{R}, U \in \mathcal{O}\mathbb{R}, V \in \mathcal{D}\mathbb{R}\}$, where $C^{UV} = \{(U, \mathbb{R}), (\mathbb{R}, V)\}$.

- L is a normal biframe.
- So $\mathcal{U}L$ is a quasi-uniformity on L.
- $(L, \mathcal{U}L)$ is obviously totally bounded.
- $C(L, \mathcal{U}L)$ is not compact.
- So $(L, \mathcal{U}L)$ does not have enough regular Cauchy bifilters.

Examples of erc quasi-nearness biframes

We recall the doubling and symmetrizing functors that relate nearness frames and quasi-nearness biframes:

$$\mathbf{NearFrm} \underset{S \text{ “symmetrizing”}}{\overset{D \text{ “doubling”}}{\rightleftarrows}} \mathbf{QNearBiFrm}$$

- For $(L, \mathcal{U}L)$ a nearness frame, $DL = (L, L, L)$ and $D\mathcal{U}L$ is generated by $\{D^d : D \in \mathcal{U}L\}$, where $D^d = \{(d, d) : d \in D\}$.
- For $(M, \mathcal{U}M)$ a quasi-nearness biframe, $SM = M_0$ and $S\mathcal{U}M$ is generated by $\{C^s : C \in \mathcal{U}M\}$, where $C^s = \{c \wedge \tilde{c} : (c, \tilde{c}) \in C\}$.

Note that $SD = \mathrm{id}_{\mathbf{NearFrm}}$.

Lemma

(a) A nearness frame $(L, \mathcal{U}L)$ is erc if and only if $D(L, \mathcal{U}L)$ is erc.

(b) If a quasi-nearness biframe $(M, \mathcal{U}M)$ is erc, then $S(M, \mathcal{U}M)$ is erc, but not conversely.

Example

This is a straightforward example of a quasi-nearness biframe that has enough regular Cauchy bifilters, for the simple reason that all bifilters on it are regular.

Let L be an extremely zero-dimensional biframe. This means that each $x \in L_1$ has a complement x' in L_0 that is a member of L_2 and similarly for $x \in L_2$. Let $\mathcal{SB}(L)$ be the standard binary quasi-nearness on L. Now for any $a \in L_1$, we have $a \lhd_1 a$, so $\varphi(a) = \bigvee\{\varphi(z) : z \lhd_1 a\}$ trivially.

Lemma

Let $(M, \mathcal{U}M)$ be a strong quasi-nearness biframe and $\varphi : M \to T$ a Cauchy bifilter such that $\varphi^\circ : M \to T$ is also a bifilter on M. Then φ° is a Cauchy bifilter.

Lemma

Suppose $(M, \mathcal{U}M)$ is a quasi-nearness biframe in which the uniformly below relation $\lhd = (\lhd_1, \lhd_2)$ interpolates.
Suppose that $\varphi : M \to T$ and $\varphi^\circ : M \to T$ are bifilters on M.
For $a \in L_i, i = 1, 2$,
if $\varphi^\circ(a) = \bigvee\{\varphi(z) : z \lhd_i a\}$, then $\varphi^\circ(a) = \bigvee\{\varphi^\circ(z) : z \lhd_i a\}$.

Example

Let X be a set linearly ordered by $\leq$.
We write $(-\infty, a) = \{x \in X : x < a\}$ and $(b, \infty) = \{x \in X : x > b\}$.
L_1 is the topology on X with base $\{(-\infty, a) : a \in X\}$.
L_2 is the topology on X with base $\{(b, \infty) : b \in X\}$.
L_0 has subbase $\{(-\infty, a) : a \in X\} \cup \{(b, \infty) : b \in X\}$.
We will call such (L_0, L_1, L_2) an order topology biframe.

Any order topology biframe with a quasi-nearness that is strong and has an interpolating uniformly below relation, has enough regular Cauchy bifilters. This clearly applies to the biframe of reals, the biframe of rationals and the biframe of integers.