Many for the price of one duality principle for variety-based topological spaces

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Outline

Duality principle for topological spaces

- Recently, D. Hofmann considered topological spaces as generalized orders, and characterized the ones, which satisfy a suitably defined topological analogue of the complete distributivity law.
- He showed that the category of distributive spaces is dually equivalent to a certain category of frames, since they both represent the idempotent split completion of the same category.
- The developments are based in four particular submonads of the filter monad on the category **Top** of topological spaces and continuous maps, providing four dualities of the same kind.

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- The developments are based in four particular submonads of the filter monad on the category **Top** of topological spaces and continuous maps, providing four dualities of the same kind.

Variety-based modification

- This talk lifts the above-mentioned dualities of D. Hofmann to the framework of lattice-valued fixed-basis topological spaces.
- We replace the variety of frames, which underlies the category **Top**, with an arbitrary one, and find the sufficient conditions on its algebras, which allow to get an analogue of the concept of distributivity of D. Hofmann as well as his obtained dualities.
- Our provided machinery gives rise to many dualities of the same kind, which, additionally, could rely on lattice-valued topology.

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- Our provided machinery gives rise to many dualities of the same kind, which, additionally, could rely on lattice-valued topology.

Variety-based topological spaces

Definition 1

Given a variety of algebras $\mathbf{A}$ and an $\mathbf{A}$-algebra A, A-**Top** is the construct, which is defined by the following data:

objects are pairs (X,τ), with X a set and τ an $\mathbf{A}$-subalgebra of A^X;

morphisms $(X_1,\tau_1) \xrightarrow{f} (X_2,\tau_2)$ are maps $X_1 \xrightarrow{f} X_2$ such that $f_A^{\leftarrow}(\alpha) = \alpha \circ f \in \tau_1$ for every $\alpha \in \tau_2$.

Example 2

If $\mathbf{A} = \mathbf{Frm}$ and $A = \mathbf{2}$, then $\mathbf{2}$-**Top** $\cong$ **Top**.

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If $\mathbf{A} = \mathbf{Frm}$ and $A = \mathbf{2}$, then $\mathbf{2}$-$\mathbf{Top} \cong \mathbf{Top}$.

Variety-based monads

Assumption: $\mathbf{B}$ is a reduct of $\mathbf{A}$ with the forgetful functor $\mathbf{A} \xrightarrow{|-|} \mathbf{B}$.

Theorem 3

1. *There exists the functor A-$\mathbf{Top} \xrightarrow{\mathcal{O}_A} \mathbf{B}^{op}$, which is defined by $\mathcal{O}_A((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = |\tau_1| \xrightarrow{(f_A^{\leftarrow})^{op}} |\tau_2|$.*
2. *There exists the functor $\mathbf{B}^{op} \xrightarrow{Pt_A} A$-$\mathbf{Top}$, which is defined by $Pt_A(B_1 \xrightarrow{\varphi} B_2) = (Pt_A(B_1), \tau_1) \xrightarrow{(\varphi^{op})_A^{\leftarrow}} (Pt_A(B_2), \tau_2)$, with $Pt_A(B_i) = \mathbf{B}(B_i, |A|)$, and τ_i the $\mathbf{A}$-algebra generated by the image of B_i under the map $B_i \xrightarrow{\Phi_A} |A^{Pt_A(B_i)}|$, $(\Phi_A(b))(p) = p(b)$.*
3. *Pt_A is a right adjoint to $\mathcal{O}_A$.*
4. *The adjunction gives rise to a monad $\mathbb{F} = (F, \eta, \mu)$ on A-$\mathbf{Top}$.*

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2. *There exists the functor* $\mathbf{B}^{op} \xrightarrow{Pt_A} A$-$\mathbf{Top}$, *which is defined by* $Pt_A(B_1 \xrightarrow{\varphi} B_2) = (Pt_A(B_1),\tau_1) \xrightarrow{(\varphi^{op})_A^{\leftarrow}} (Pt_A(B_2),\tau_2)$, *with* $Pt_A(B_i) = \mathbf{B}(B_i, |A|)$, *and* τ_i *the* **A**-*algebra generated by the image of* B_i *under the map* $B_i \xrightarrow{\Phi_A} |A^{Pt_A(B_i)}|$, $(\Phi_A(b))(p) = p(b)$.
3. Pt_A *is a right adjoint to* $\mathcal{O}_A$.
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Examples of variety-based monads

Example 4

If $\mathbf{A} = \mathbf{Frm}$ and $A = \mathbf{2}$, then

1. $\mathbf{B} = \mathbf{Frm}$ provides the completely prime filter monad;
2. $\mathbf{B} = \mathbf{BLat}$ provides the prime filter monad;
3. $\mathbf{B} = \mathbf{SLat}(\wedge, \top)$ provides the filter monad;
4. $\mathbf{B} = \mathbf{BSLat}(\wedge)$ provides the proper filter monad;
5. $\mathbf{B} = \mathbf{BSLat}(\wedge, \bigvee_d)$ provides the Scott-open filter monad.

Variety-based T_0 spaces

Definition 5

1. A space (X, τ) is said to be T_0 provided that for every distinct $x_1, x_2 \in X$ there exists $\alpha \in \tau$ such that $\alpha(x_1) \neq \alpha(x_2)$.
2. A-$\mathbf{Top}_0$ is the full subcategory of A-$\mathbf{Top}$ of T_0 spaces.

Assumption: $\mathbf{B}$ has a forgetful functor to $\mathbf{Pos}$.

Theorem 6

There exists the functor A-$\mathbf{Top}_0 \xrightarrow{Spec} \mathbf{Pos}$, *which is given by* $Spec((X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)) = (X_1, \sqsubseteq) \xrightarrow{f} (X_2, \sqsubseteq)$, *where* $x \sqsubseteq y$ *iff* $\alpha(y) \leqslant \alpha(x)$ *for every* $\alpha \in \tau_i$.

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Kock-Zöberlein variety-based monads

Theorem 7

There exists the restriction $\mathbb{F}_0$ of the monad $\mathbb{F}$ to A-$\mathbf{Top}_0$.

Assumption: For every $A \in \mathbf{A}$, $\leqslant$ is a subalgebra of $A \times A$.

Assumption: Given a set X, for every $x \in X$ and every $\mathbf{B}$-algebra $S \subseteq A^X$: if $\alpha \in \langle S \rangle$ and $\alpha(x) \neq \perp$, then there exists $s \in S$ such that $s \leqslant \alpha$ and $s(x) = \alpha(x)$.

Theorem 8

The monad $\mathbb{F}_0$ is of Kock-Zöberlein type.

Corollary 9

The $\mathbb{F}_0$-algebra structure on a T_0 space (X, τ) is unique.

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Variety-based monad morphisms

Theorem 10

Suppose the diagram

$$\begin{tikzcd} & \mathbf{A} \arrow[dl, "|-|"'] \arrow[dr, "|-|"] & \\ \mathbf{B}_1 \arrow[rr, "|-|"'] & & \mathbf{B}_2 \end{tikzcd}$$

commutes. Then there exists a monad morphism $\mathbb{F}^1 \xrightarrow{\xi} \mathbb{F}^2$, *which is defined by the inclusions* $Pt_A^1(|\tau|) \hookrightarrow Pt_A^2(|\tau|)$.

Corollary 11

There exists a functor $(A\text{-}\mathbf{Top}_0)^{\mathbb{F}_0^2} \xrightarrow{G} (A\text{-}\mathbf{Top}_0)^{\mathbb{F}_0^1}$.

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Sobriety for variety-based monads

Definition 12

A T_0 space (X, τ) is said to be $|A|$-sober (A-sober in case of $\mathbf{A} = \mathbf{B}$) provided that the map $(X, \tau) \xrightarrow{\eta_{(X,\tau)}} F_0(X, \tau)$ is a homeomorphism.

Theorem 13

Every $\mathbb{F}_0$-algebra (X, τ) is A-sober. If $\mathbf{A} = \mathbf{B}$, then every A-sober T_0 space (X, τ) is an $\mathbb{F}_0$-algebra.

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Characterization of monad algebras I

Definition 14

Let (X,τ) be a T_0 space.

1. Given $p \in Pt_A(|\tau|)$, define $\lim(p) = \{x \in X \mid \eta_{(X,\tau)}(x) \leqslant p\}$.
2. Given $\alpha, \beta \in \tau$, α is said to be $\mathbb{F}_0$-below β (denoted $\alpha \ll_{\mathbb{F}_0} \beta$) provided that for every $p \in Pt_A(|\tau|)$, there exists $x \in \lim(p)$ such that $p(\alpha) \leqslant \beta(x)$.
3. (X,τ) is said to be $\mathbb{F}_0$-core-compact provided that for every $\beta \in \tau$ and every $x \in X$ such that $\beta(x) \neq \bot$, there exists $\alpha \in \tau$ such that $\beta(x) \leqslant \alpha(x)$ and $\alpha \ll_{\mathbb{F}_0} \beta$.
4. (X,τ) is said to be $\mathbb{F}_0$-stable provided that for every $p \in Pt_A(|\tau|)$, there exists $p' \in Pt_A(\tau)$ such that $p' \leqslant p$ and $\lim(p') = \lim(p)$.

Characterization of monad algebras II

Assumption: Let X be a set and let $S \subseteq A^X$ be a $\mathbf{B}$-algebra. If $\alpha \in A^X$ has the property that for every $x \in X$ such that $\alpha(x) \neq \perp$, there exists $s \in S$ such that $s \leqslant \alpha$ and $s(x) = \alpha(x)$, then $\alpha \in \langle S \rangle$.

Theorem 15

Given a T_0 space (X, τ), the following are equivalent:

1. *(X, τ) is an $\mathbb{F}_0$-algebra;*
2. *(X, τ) is A-sober, $\mathbb{F}_0$-stable and $\mathbb{F}_0$-core-compact.*

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Characterization of distributive monad algebras I

Definition 16

An $\mathbb{F}_0$-algebra $((X,\tau),h)$ is said to be $\mathbb{F}_0$-distributive provided that h has a left adjoint $(X,\tau)\xrightarrow{t}F_0(X,\tau)$ in A-$\mathbf{Top}_0$.

Theorem 17

An $\mathbb{F}_0$-algebra $((X,\tau),h)$ is $\mathbb{F}_0$-distributive if and only if there exists $(X,\tau)\xrightarrow{t}F_0(X,\tau)$ in $(A\text{-}\mathbf{Top}_0)^{\mathbb{F}_0}$ such that $h\circ t=1_{(X,\tau)}$.

Definition 18

An $\mathbb{F}_0$-algebra $((X,\tau),h)$ is said to be $\mathbb{F}_0$-disconnected provided that given $\alpha\in\tau$, for every $x\in X$, there exists $\max\{p(\alpha)\,|\,h(p)=x\}$ (denoted $(\mu(\alpha))(x)$), and, moreover, $\mu(\alpha)\in\tau$.

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Characterization of distributive algebras II

Assumption: F_0 takes surjective maps to surjective maps.

Assumption: Given a T_0 space (X, τ), for every $a \in A$ and every $\alpha \in \tau$, $(F_0 i)^{\rightarrow}(F_0(X^a_\alpha, \hat{\tau})) = (F_0(X, \tau))^a_\alpha$, where

$$X^a_\alpha = \{x \in X \mid a \leqslant \alpha(x)\} \xhookrightarrow{i} X$$
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Characterization of distributive algebras III

Assumption: Let X be a set and let $|A^X| \xrightarrow{f} |A^X|$ be a map. Given $\lambda \in \Lambda_{\mathbf{B}}$, $(\alpha_i)_{n_\lambda} \in (A^X)^{n_\lambda}$ and $S \subseteq n_\lambda$, define

$$\overline{\alpha_i}^S = \begin{cases} \alpha_i, & i \in S \\ f(\alpha_i), & i \notin S. \end{cases}$$

If $\omega_\lambda^{A^X}((f(\alpha_i))_{n_\lambda}) \leqslant f(\omega_\lambda^{A^X}((\overline{\alpha_i}^S)_{n_\lambda}))$ for every finite $S \subseteq n_\lambda$, then it follows that $\omega_\lambda^{A^X}((f(\alpha_i))_{n_\lambda}) \leqslant f(\omega_\lambda^{A^X}((\alpha_i)_{n_\lambda}))$.

Characterization of distributive algebras IV

Theorem 19

Given an $\mathbb{F}_0$-algebra (X, τ), the following are equivalent:

1. (X, τ) *is* $\mathbb{F}_0$*-distributive;*
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Variety-based duality I

Definition 20

$\mathrm{spl}((A\text{-}\mathbf{Top}_0)^{\mathbb{F}_0})$ is the full subcategory of $(A\text{-}\mathbf{Top}_0)^{\mathbb{F}_0}$ of $\mathbb{F}_0$-distributive $\mathbb{F}_0$-algebras.

Theorem 21

Idempotents split in $A\text{-}\mathbf{Top}_0$.

Corollary 22

$\mathrm{spl}((A\text{-}\mathbf{Top}_0)^{\mathbb{F}_0}) \simeq \mathrm{kar}((A\text{-}\mathbf{Top}_0)_{\mathbb{F}_0})$.

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Variety-based duality II

Assumption: The variety **A** has a nullary operation.

Theorem 23

There exists a full embedding $(A\text{-}\mathbf{Top}_0)_{\mathbb{F}_0} \xrightarrow{L} \mathbf{B}^{op}$.

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$\mathbf{B}^{op}_{\mathrm{kar}}$ is the idempotent split completion of the image of $(A\text{-}\mathbf{Top}_0)_{\mathbb{F}_0}$ in $\mathbf{B}^{op}$ under L.

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Variety-based frames I

Theorem 26

$B \in \mathbf{B}^{op}_{\mathrm{kar}}$ if and only if there exists a T_0 space (X, τ) such that B is a retract of $|\tau|$.

Definition 27

1. $B \in \mathbf{B}$ is A-spatial provided that for every $b_1, b_2 \in B$ such that $b_1 \not\leqslant b_2$, there exists $p \in Pt_A(B)$ such that $p(b_1) \not\leqslant p(b_2)$.
2. $A \in \mathbf{A}$ is a $\mathbf{B}$-frame provided that A has a reduct in **Sup**, and, moreover, for every $\lambda \in \Lambda_{\mathbf{B}}$ such that $n_\lambda \neq 0$, and every family $\{S_i \subseteq A \mid i \in n_\lambda\}$, $\omega^A_\lambda((\bigvee S_i)_{n_\lambda}) = \bigvee\{\omega^A_\lambda((s_i)_{n_\lambda}) \mid s_i \in S_i, i \in n_\lambda\}$.

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Definition 27

1. $B \in \mathbf{B}$ is *A-spatial* provided that for every $b_1, b_2 \in B$ such that $b_1 \not\leqslant b_2$, there exists $p \in Pt_A(B)$ such that $p(b_1) \not\leqslant p(b_2)$.
2. $A \in \mathbf{A}$ is a **B**-frame provided that A has a reduct in **Sup**, and, moreover, for every $\lambda \in \Lambda_{\mathbf{B}}$ such that $n_\lambda \neq 0$, and every family $\{S_i \subseteq A \mid i \in n_\lambda\}$, $\omega^A_\lambda((\bigvee S_i)_{n_\lambda}) = \bigvee \{\omega^A_\lambda((s_i)_{n_\lambda}) \mid s_i \in S_i, i \in n_\lambda\}$.

Variety-based frames II

Assumption: A is a **B**-frame.

Assumption: Let $S \subseteq A$ be a **B**-algebra, let $\lambda \in \Lambda_{\mathbf{B}}$, and let $(a_i)_{n_\lambda} \in \langle S \rangle^{n_\lambda}$ have the property that $a_i \neq \perp$ for every $i \in n_\lambda$. Given $s \in S$ such that $s \leqslant \omega_\lambda^{\langle S \rangle}((a_i)_{n_\lambda})$, there exists $(s_i)_{n_\lambda} \in S^{n_\lambda}$ such that $s_i \leqslant a_i$ for every $i \in n_\lambda$, and $s \leqslant \omega_\lambda^{\langle S \rangle}((s_i)_{n_\lambda})$.

Variety-based frames II

Assumption: A is a $\mathbf{B}$-frame.

Assumption: Let $S \subseteq A$ be a $\mathbf{B}$-algebra, let $\lambda \in \Lambda_{\mathbf{B}}$, and let $(a_i)_{n_\lambda} \in \langle S \rangle^{n_\lambda}$ have the property that $a_i \neq \bot$ for every $i \in n_\lambda$. Given $s \in S$ such that $s \leqslant \omega_\lambda^{\langle S \rangle}((a_i)_{n_\lambda})$, there exists $(s_i)_{n_\lambda} \in S^{n_\lambda}$ such that $s_i \leqslant a_i$ for every $i \in n_\lambda$, and $s \leqslant \omega_\lambda^{\langle S \rangle}((s_i)_{n_\lambda})$.

Variety-based frames III

Definition 28

B-Frm is the full subcategory of **B** of A-spatial **B**-frames.

Theorem 29

$(\mathbf{B}\text{-}\mathbf{Frm})^{op} \cong \mathbf{B}^{op}_{\mathrm{kar}}$.

Corollary 30

$(\mathbf{B}\text{-}\mathbf{Frm})^{op} \simeq \mathrm{spl}((A\text{-}\mathbf{Top}_0)^{\mathbb{F}_0})$.

Example 31

If $\mathbf{A} = \mathbf{B}$, then **A-Frm** is the category of A-spatial **A**-algebras, and $\mathrm{spl}((A\text{-}\mathbf{Top}_0)^{\mathbb{F}_0})$ is the category of A-sober T_0 topological spaces.

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Conclusion

- In the talk, we have provided a variety-based approach to the four dualities of D. Hofmann.
- Our approach is based in a series of assumptions, which are sufficient to get a similar kind duality.
- Every variety, which satisfies these assumptions, will qualify, thereby providing many possible dualities.
- It will be our future work to find also the necessary conditions for the obtained machinery.

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Thank you for your attention!