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Frames and frame relations

From independent discoveries (Bruns and Lakser [1]; Horn and Kimura [2]), we know that frames are precisely the injective (meet) semilattices. Simply knowing this does not get us very far in studying frames *qua* frames. But the full subcategory of SL consisting of injective semilattices comes closer. A semilattice homomorphism $h\colon M \to L$ between two frames can be viewed "approximably" as a relation $R_h \subseteq L \times M$ defined by $x \leq h(y)$. Such a relation is closed under weakening: $x \leq x'$, $x' R_h y'$ and $y' \leq y$ implies $x R_h y$. It is a subframe of $L \times M$. Any such relation, called a frame relation, determines a semilattice homomorphism. So the category $\overline{\mathsf{Frm}}$ of frames and frame relations is opposite to the full subcategory of SL consisting of injective semilattices.

In $\overline{\mathsf{Frm}}$, an adjoint pair of morphisms, i.e., an adjoint pair of frame relations corresponds to an adjoint pair of semilattice homomorphisms between injective semilattives. So by adjunction, the lower adjoint determines a semilattice homomorphism preserving all joins. Thus the category Map(Frm), consisting of frames and adjoint pairs of frame relations $(\check{f} \dashv \hat{f})$, is isomorphic to the category Frm.

A subobject (a sublocale) is then an isomorphism class of extremal epimorphisms in this category. One can check that a sublocale of a frame corresponds exactly to an idempotent, reflexive frame relation.

What we have said so far is mostly known, at least in folklore. Nevertheless, this points toward studying frames directly via frame relations, as they are (dual to) the natural morphisms of injective semilattices. To illustrate the point, we construct the assembly of a frame via frame relations, showing that its familiar universal property comes about naturally from this approach.

References

[1] Bruns, G. and Lakser, H. Injective hulls of semilattices. *Canadian Mathematics Bulletin* 13 (1970) 115–118.

[2] Horn, A. and Kimura, N. The category of semilattices. *Algebra Universalis* 1 (1971) 26–38.

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