Frames and Frame Relations

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The Next Poem — Dana Gioia

How much better it seems now
than when it is finally done –
the unforgettable first line,
the cunning way the stanzas run.

The rhymes soft-spoken and suggestive
are barely audible at first,
an appetite not yet acknowledged
like the inkling of a thirst.

While gradually the form appears
as each line is coaxed aloud –
the architecture of a room
seen from the middle of a crowd.

The music that of common speech
but slanted so that each detail
sounds unexpected as a sharp
inserted in a simple scale.

No jumble box of imagery
dumped glumly in the readers lap
or elegantly packaged junk
the unsuspecting must unwrap.

But words that could direct a friend
precisely to an unknown place,
those few unshakeable details
that no confusion can erase.

And the real subject left unspoken
but unmistakable to those
who dont expect a jungle parrot
in the black and white of prose.

How much better it seems now
than when it is finally written.
How hungrily one waits to feel
the bright lure seized, the old hook bitten.

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- Completely distributive lattices as a starting point

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In particular,

- The assembly of a frame comes about as a sublocale $\mathcal{Q}(L)$ of a particular completely distributive lattice.
- Proof that $\mathcal{Q}(L)$ has the universal property of the assembly using simple combinatorial reasoning – essentially via a kind of sequent calculus.

First step: Weakening Relations

Definition

For posets A and B, a weakening relation is a relation $R \subseteq A \times B$ so that

$$\frac{x \leq_X x' \; R \; y' \leq_Y y}{x \; R \; y}$$

We denote this by $R\colon X \looparrowright Y$.
Pos will denote the category of posets and weakening relations.

- id_X is simply $\leq_X$.
- Composition is relational product (but I write $R; S$ instead of $S \circ R$.

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- A w. relation $R\colon A \looparrowright B$ satisfies $\text{id}_A \subseteq (\text{id}_B/R); R$ if and only if it is determined by a monotone function $f\colon A \to B$ by $x\ R\ y$ iff $f(x) \leq y$.

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- If A has binary meets and B has binary joins, Heyting arrows in $\overline{\mathrm{Pos}}(A, B)$ are defined by

$$\frac{\forall x, y . x \, R \, y \Rightarrow x \wedge a \, S \, b \vee x}{a \, (R {\to} S) \, b}$$

Meet and Sup Stability

Definition

- If B is a (unital) meet semilattice, say $R\colon A \nrightarrow B$ is meet-stable if

$$\frac{}{x \mathrel{R} 1} \qquad \frac{x \mathrel{R} y \qquad x \mathrel{R} y'}{x \mathrel{R} y \wedge y'}$$

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- $\overline{\text{SLat}}$: category of meet semilattices with meet stable relations
  $\overline{\text{Sup}}$: category of sup lattices with sup-stable relations.
  $\overline{\text{Frm}}$: category of frames with meet-sup-stable relations.

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- Clearly stable relations (either kind) are closed under intersection.
- We then use our nice characterization of Heyting arrow to check that if S is stable, so is $R \to S$.
- $\overline{\mathrm{Frm}}(A, B) = \overline{\mathrm{SLat}}(A, B) \cap \overline{\mathrm{Sup}}(A, B)$.

Fruit Requiring a Small Step Stool

Lemma

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- Using this, check that

$$R; F \subseteq F; S$$

is a frame relation from $\overline{\mathrm{Frm}}(A, A)$ to $\overline{\mathrm{Frm}}(B, B)$ that has an adjoint. So it determines a frame homomorphism.

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- Checking that this respects identity and composition is easy.

Picking Things We Dropped on the Ground

Definition

Let $\mathcal{E}(A) = \overline{\mathsf{Frm}}(A, A)$.
Let $\mathcal{R}(A) =$ the closed sublocale of $\mathcal{E}(A)$ determined by id_A.
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Easy to check

- $\mathcal{R}$ is also functorial – exactly as $\mathcal{E}$.
- Define relations $\gamma_a, \upsilon_a \in \mathcal{R}(A)$.

$$\frac{x \leq a \vee y}{x \gamma_a y} \qquad \frac{x \wedge a \leq y}{x \upsilon_a y}$$

These not only contain id_A, but are transitive relations.

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5. *The frame relation $\Gamma\colon A \looparrowright A$ defined by $\gamma_a \subseteq R$ satisfies* $id_A \subseteq \Gamma; (id_{\mathcal{A}}/\Gamma)$.

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Proof:

(1) and (2) are now routine. ...

Proof continued

We need this

Lemma

A frame relation $R \in \mathcal{R}(A)$ is transitive iff it admits Gentzen's Cut:

$$\frac{u \mathrel{R} v \vee w \qquad w \wedge x \mathrel{R} y}{u \wedge x \mathrel{R} v \vee y}$$

"If" is easy. "Only if"

$$\frac{\dfrac{u \mathrel{R} v \vee w \qquad x \mathrel{R} x}{u \wedge x \mathrel{R} (v \vee w) \wedge x} \qquad \dfrac{w \wedge x \mathrel{R} y \qquad v \mathrel{R} v}{v \vee (w \wedge x) \mathrel{R} v \vee y}}{u \wedge x \mathrel{R} v \vee y}$$

Proof continued

- $$\frac{x\gamma_a a \qquad a\upsilon_a y}{x\gamma_a;\upsilon_a y}$$

  But any transitive relation containing γ_a and υ_a contains $\gamma_a;\upsilon_a$.

- For any a, b, $a\gamma_a b$ and $a\upsilon_b b$. So $R \subseteq \bigcup_{aRb}(\gamma_a \cap \upsilon_b)$. And suppose $a\ R\ b$ and $x(\gamma_a \cap \upsilon_b)y$. Then

$$\frac{\dfrac{x\ R\ y \vee a \qquad a\ R\ b}{x\ R\ y \vee b} \qquad b \wedge x\ R\ y}{x\ R\ y}$$

Proof continued

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  But any transitive relation containing γ_a and υ_a contains $\gamma_a;\upsilon_a$.
- If $x\gamma_a y$ and $x\upsilon_a y$, then
  $$\frac{x \leq y \vee a \qquad a \wedge x \leq y}{x \leq y}$$
  So $\gamma_a \cap \upsilon_a = \mathrm{id}_A$.
- For any a, b, $a\gamma_a b$ and $a\upsilon_b b$. So $R \subseteq \bigcup_{aRb}(\gamma_a \cap \upsilon_b)$. And suppose $a\ R\ b$ and $x(\gamma_a \cap \upsilon_b)y$. Then
  $$\frac{\dfrac{x\ R\ y \vee a \qquad a\ R\ b}{x\ R\ y \vee b} \qquad b \wedge x\ R\ y}{x\ R\ y}$$

Definitely Need a Proper Ladder Now

Definition

Define on any frame B, $\prec_B : B \nrightarrow B$ by

$$\frac{w \wedge x \leq 0 \qquad 1 \leq y \vee w}{x \prec_B y}$$

This is meet stable, not sup stable.

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Theorem

For any frame A, $\Gamma \colon A \nrightarrow \mathcal{Q}(A)$ is universal with respect to functional frame relations for $R \colon A \nrightarrow B$ satisfying

1. $id_A \subseteq R; (id_B/R)$;
2. $(id_B/R); R \subseteq \prec_B$.

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Proof.

Not in a 30 minute talk!

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- Inside that, one finds the dense sub-identities: $R \subseteq R; R$ and $R \subseteq \mathrm{id}_A$.
- The dense sub-identities correspond exactly to subframes of A.
- So $\mathcal{E}(A)$ is a frame in which all sublocales (transitive relations containing id_A) and all subframes (dense relations contained in id_A) reside.

Happy Birthday, Ales.
And thank you for your next ~~poem~~ theorem.